Pluripotential Operadic Calculus

by

Anna Sopena i Gilboy

A thesis submitted for the PhD programme
in Mathematics and Computer Science

Advisors:
Joana Cirici
Jonas Stelzig

Departament de Matemàtiques i Informàtica
Universitat de Barcelona
Barcelona, May 2026

Memòria presentada per aspirar al grau de Doctor en Matemàtiques i Informàtica per la Universitat de Barcelona.

Certifico que aquesta memòria ha estat realitzada per en Anna Sopena i Gilboy, sota la meva direcció.
Barcelona, 25 de juny de 2026

Joana Cirici Jonas Stelzig

Als meus pares

Abstract

This thesis explores a new context for operadic calculus adapted to the study of complex manifolds or, more abstractly, of bidifferential bigraded algebras up to pluripotential weak equivalences. These are equivalences governed by Bott-Chern and Aeppli cohomologies.

In the first chapter we study the symmetric monoidal model category of bicomplexes with pluripotential weak equivalences. We introduce a functor from cochain complexes to bicomplexes, called the *inflation functor*, which sends quasi-isomorphisms to pluripotential weak equivalences. We show this functor is part of a Quillen adjunction. Its right adjoint is a well-known construction in complex geometry, which gives a sheaf-theoretic presentation of Bott-Chern and Aeppli cohomologies. We prove this functor is symmetric monoidal.

In Chapter 2, we develop pluripotential operadic theory. In particular, we compute minimal models of operads with respect to pluripotential weak equivalences. We present two methods: the first constructs minimal models inductively via free extensions, while the second uses the inflation functor to obtain minimal models through Koszul duality. These methods allow us to define *pluripotential A_∞-algebras*, i.e., algebras over the minimal model of the associative operad. We then establish a pluripotential homotopy transfer theorem for bidifferential bigraded algebras. Moreover, we prove commutative, unital, and cyclic versions of this homotopy transfer theorem.

In Chapter 3, we test the previous constructions in the study of complex manifolds. First, we show that the homotopy transfer theorem for bidifferential bigraded algebras provides a homotopical framework for the theory of Massey products in Bott-Chern and Aeppli cohomologies, defined for any complex manifold. In the case of compact Kähler manifolds, we use Hodge theory to produce canonical pluripotential A_∞-structures on the space of harmonic forms and use these to obtain results of strong formality, a notion of formality naturally arising when considering pluripotential weak equivalences.

Resum

Aquesta tesi explora un nou context per al càlcul operàdic adaptat a l'estudi de varietats complexes o, de manera més abstracta, d'àlgebres bigraduades bidiferencials llevat d'equivalència feble pluripotencial. Aquestes són equivalències governades per les cohomologies de Bott-Chern i d'Aeppli.

En el primer capítol estudiem la categoria de models monoidal simètrica de bicomplexos amb equivalències febles pluripotencials. Introduïm un functor de la categoria de complexos de cocadenes a la categoria de bicomplexos, anomenat *inflation functor*, que envia quasiisomorfismes a equivalències febles pluripotencials. Demostrem que aquest functor forma part d'una adjunció de Quillen. El seu adjunt per la dreta és una construcció ben coneguda en geometria complexa, que proporciona una descripció en termes de feixos de les cohomologies de Bott-Chern i d'Aeppli. Finalment, provem que aquest functor és monoidal simètric.

En el capítol 2, desenvolupem la teoria operàdica pluripotencial. En particular, calculem models minimals d'opèrades respecte a les equivalències febles pluripotencials. Presentem dos mètodes: el primer construeix models minimals de manera inductiva mitjançant extensions lliures, mentre que el segon utilitza l'inflation functor per obtenir models minimals mitjançant la dualitat de Koszul. Aquests mètodes ens permeten definir *àlgebres A_∞ pluripotencials*, és a dir, àlgebres sobre el model minimal de l'opèrada associativa. A continuació, establim un teorema de transferència homotòpica pluripotencial per a àlgebres bigraduades bidiferencials. A més, també provem versions d'aquest teorema de transferència homotòpica per a àlgebres commutatives, unitàries i cícliques.

En el capítol 3, apliquem les construccions anteriors a l'estudi de varietats complexes. En primer lloc, mostrem que el teorema de transferència homotòpica per a àlgebres bigraduades bidiferencials proporciona un marc homotòpic per a la teoria dels productes de Massey en les cohomologies de Bott-Chern i d'Aeppli, definits per a qualsevol varietat complexa. En el cas de les varietats Kähler compactes, utilitzem la teoria d'Hodge per construir estructures A_∞ pluripotencials sobre l'espai de formes harmòniques, i les fem servir per obtenir resultats de formalitat forta, una noció de formalitat que apareix de manera natural en considerar equivalències febles pluripotencials.

Agraïments

En primer lloc, vull agrair a la Joana la seva dedicació i suport incondicional, la seva direcció tant a nivell matemàtic com a nivell personal, ensenyant-me així a navegar pel món acadèmic. Durant aquests quatre anys de doctorat n'he après moltíssim d'ella i, al mateix temps, crec que ens ho hem passat genial.

M'agradaria agrair també al Jonas la seva direcció i disposició a discutir i comentar el que fes falta. De la mateixa manera, n'he après molt parlant amb ell.

Li agraeixo al Bruno Vallette i a la Muriel Livernet el seu temps i les ganes que posaven a les nostres discussions matemàtiques durant el temps que vaig ser a París. Alhora, al Ricardo Campos i al Geoffroy Horel les converses als coffee breaks dels congressos.

Als meus companys de doctorat per les bones estones durant els dinars a les 13:15 h. En especial al David, per tot el suport, la seva paciència i per estar sempre disposat i a punt per a, o bé xafardejar, o bé compartir el seu ampli coneixement en categories d'ordre superior i Latex. També al Pedro, per la seva companyia i per haver-me proposat treballar junts. Igualment, m'ha fet aprendre i n'he tret molt profit i, cal dir-ho, la vida és més lleugera quan arriba l'inevitable moment en què t'adones que el Lema està malament, però, aquesta vegada, l'error és compartit.

Als meus pares i a la Maite per cuidar-me tant i fer-me saber que estan orgullosos de mi.

Als meus amics, els sepsis, per ser els meus amics. No, no és *soso*, ho dic en un sentit veritablement profund, tranquils.

A la Maria, per estimar-me i cuidar-me, i per la seva bona actitud quan intento explicar-li per què $1 + 1 = 0$ a $\mathbb{Z}_2$, la infinitud dels nombres primers o el teorema de la bola peluda.

Contents

Introduction

On a smooth manifold, we can consider several additional geometric structures that refine known topological invariants and, in some situations, restrict the topology of the manifold.

A primary example is that of an *almost complex structure* on a smooth manifold M. This is an endomorphism $J : TM \to TM$ of the tangent bundle such that $J^2 = -\mathrm{id}$. This mimics multiplication by $\sqrt{-1}$ and gives the manifold a structure locally resembling a complex vector space, though not necessarily arising from complex coordinates. An almost complex structure J is said to be *integrable* when its Nijenhuis tensor vanishes. By the Newlander–Nirenberg Theorem, this condition is equivalent to the existence of a holomorphic atlas inducing J, in which case M is said to be a *complex manifold*. Almost complex structures force the manifold to be even-dimensional and orientable. In fact, the obstructions for a compact manifold M to admit an almost complex structure are purely topological. They are measured by characteristic classes and are fairly well understood. However, the boundary between almost complex and complex manifolds remains mysterious: in real dimension two, every almost complex structure is integrable. In dimension four, there are examples of almost complex manifolds not admitting any integrable structure. The main gap appears in real dimension ≥ 6, for which there is no known example of an almost complex manifold not admitting any integrable structure. Yau [Yau24] conjectured that such examples should not exist.

A related geometric structure, arising from a different but compatible viewpoint, is that of a *symplectic form*. This is a closed, nondegenerate differential 2-form ω on the manifold. Every symplectic manifold admits a compatible almost complex structure J, in the sense that $\omega(., J.)$ defines a Riemannian metric. The converse is not true, as evidenced by direct topological obstructions. For instance, even Betti numbers of compact symplectic manifolds are always non-zero, as the symplectic form represents a non-trivial cohomology class $[\omega] \in H^2(M;\mathbb{R})$ whose powers $[\omega^k] \in H^{2k}(M;\mathbb{R})$, for $2k \leq \dim M$, are also non-trivial. Other strong topological constraints arise from the theory of pseudoholomorphic curves and gauge-theoretic tools [Gom01].

When the compatible almost complex structure J on a symplectic manifold (M, ω) is integrable, we obtain a *Kähler manifold*. Compact Kähler manifolds satisfy strong cohomological constraints, including the Hodge symmetries and the hard Lefschetz theorem, leading to a far more rigid topology than that of general complex or symplectic manifolds. In particular, all odd-degree Betti numbers are even. From a homotopical perspective, the Formality Theorem of Deligne, Griffiths, Morgan, and Sullivan [DGMS75] asserts that the rational homotopy type of any compact Kähler manifold is entirely determined by its cohomology ring.

Beyond the restrictive Kähler setting, the geometry of complex manifolds becomes more intricate and less structured. One approach to studying non-Kähler manifolds is to endow them with metrics weaker than Kähler, such as balanced, Vaisman, Gauduchon, or strong Kähler with torsion (SKT) metrics, as well as related Kähler-type conditions, in an attempt to extend properties from the Kähler setting to a wider context.

The preceding overview motivates the following general problem at the intersection of Algebraic Topology and Geometry:

How do geometric structures reflect on the homotopy type of a manifold?

A very first stage to address the above question consists in transforming geometric data into algebraic data in order to refine cohomological and homotopical invariants. The resulting refined invariants can then lead to two further directions: on the one hand, they can be used in the classification of geometric structures; on the other hand, such homotopical refinements can yield new topological obstructions to the existence of certain geometric structures.

In this thesis, we address this question in the case of complex structures, focusing on their real homotopy types, which are encoded by the de Rham algebra of the underlying smooth manifold. Evidently, we are not the first to think about this particular problem, and there are multiple results in the literature in this direction. In particular, there exist refinements both at the level of cohomology and at the level of rational homotopy types, adapted to the complex manifold setting that we review below.

A common starting point in all these refinements is the decomposition

$$\mathcal{A}^n_{\mathrm{dR}}(M;\mathbb{C}) := \mathcal{A}_{\mathrm{dR}}(M)\otimes\mathbb{C} = \bigoplus_{p+q=n} A^{p,q},$$

of the complexified de Rham algebra of differential forms of a complex manifold M. Under this decomposition, the exterior differential splits into two components $d=\partial+\bar\partial$ of bidegree $(1,0)$ and $(0,1)$, respectively. The fact that $d^2=0$ implies $\partial^2=0$, $\bar\partial^2=0$, and $\partial\bar\partial=-\bar\partial\partial$. In particular, we obtain two anti-commuting differentials, forming a *bicomplex* (also called *double complex* in the literature). From such an object we can compute several cohomological invariants. The first obvious and most classical is the de Rham cohomology, so the cohomology with respect to the total differential d. We can also consider cohomology with respect to one of the operators ∂ and $\bar\partial$. These are symmetric viewpoints with respect to complex conjugation, but the connection with holomorphic forms has historically shifted the balance towards the study of $\bar\partial$-cohomology $H^{*,*}_{\bar\partial}(M)$, called *Dolbeault cohomology.* Dolbeault proved a holomorphic version of the de Rham theorem in the complex setting, showing that Dolbeault cohomology is isomorphic to the sheaf cohomology of holomorphic p-forms [Dol53]. This analytic invariant is related to the topology of the manifold through the *Frölicher spectral sequence.* The latter is the spectral sequence associated to the Hodge filtration, defined as the column filtration on the decomposition of complex forms:

$$F^p\mathcal{A}^*_{\mathrm{dR}}(M;\mathbb{C}) = \bigoplus_{j\geq p} A^{j,*}.$$

It satisfies $E_0^{*,*}=(A^{*,*},\bar\partial)$ and $E_1^{*,*}=H^{*,*}_{\bar\partial}(M)$. Moreover, it converges to the complex de Rham cohomology $H_{\mathrm{dR}}(M;\mathbb{C})$. In particular, this spectral sequence interpolates between complex analytic information and purely topological information. Its E_∞-page may be viewed as a refinement of de Rham cohomology.

Beyond the Frölicher spectral sequence, the bicomplex structure allows for two further natural cohomology theories: *Bott-Chern* [BC65] and *Aeppli* [Aep65] cohomologies, defined by

$$H^{*,*}_{BC}(M) := \frac{\mathrm{Ker}\partial \cap \mathrm{Ker}\bar{\partial}}{\mathrm{Im}(\partial\bar{\partial})} \quad \text{and} \quad H^{*,*}_{\mathcal{A}}(M) := \frac{\mathrm{Ker}(\partial\bar{\partial})}{\mathrm{Im}\partial + \mathrm{Im}\bar{\partial}}.$$

Bott-Chern and Aeppli cohomologies are especially interesting biholomorphic invariants for non-Kähler manifolds. By definition, these cohomologies are closely related to metrics such as balanced, Gauduchon, and SKT metrics, mentioned above.

The identity map induces natural transformations between these cohomology theories as depicted in the following diagram

$$\begin{tikzcd}
 & H^{**}_{BC} \arrow[dl] \arrow[d] \arrow[dr] & \\
H^{**}_{\partial} \arrow[r, Rightarrow] \arrow[dr] & H^{*}_{\mathrm{dR}} \arrow[d] & H^{**}_{\bar{\partial}} \arrow[l, Rightarrow] \arrow[dl] \\
 & H^{**}_{\mathcal{A}} &
\end{tikzcd}$$

where the horizontal double arrows denote the row and column spectral sequences mentioned before, the Frölicher spectral sequence and its symmetric counterpart. When all these maps are isomorphisms, the manifold is said to satisfy the *$\partial\bar{\partial}$-property.* The Formality Theorem for compact Kähler manifolds is proven through this property.

Since both differentials ∂ and $\bar{\partial}$ are derivations for the wedge product, the complexified de Rham algebra becomes a *commutative bidifferential bigraded algebra* (cbba for short). These are the objects we aim to study as a refinement of real homotopy types. In order to develop a homotopy theory for this refinement, a next step is to choose a suitable class of weak equivalences.

In view of the above diagram, a first obvious choice is to study cbba's with respect to weak equivalences governed by the middle row. There is classical and significant work done in this direction: Neisendorfer and Taylor [NT78] considered the class of weak equivalences given by maps of cbba's inducing isomorphisms in Dolbeault cohomology and developed a Dolbeault version of minimal models à la Sullivan, leading to *Dolbeault homotopy theory.* More generally, Halperin and Tanré [HT90] developed a theory of filtered minimal models, starting with a filtered algebra and with the class of weak equivalences given by maps inducing isomorphisms at a given page of the associated spectral sequence. The theory gives a spectral sequence at the level of rational homotopy theory, connecting Dolbeault and de Rham homotopy groups.

Recent developments in the study of complex manifolds are related to the consideration of the class of *pluripotential weak equivalences* [MS24], [Ste25a], [CLW25], [CGVSG25]. These are defined by those maps of cbba's inducing isomorphisms in both Bott–Chern and Aeppli cohomologies, and are the strongest ones one can extract from the above diagram while keeping the homotopical flavor. In [Ste25a], Stelzig develops a theory of minimal models à la Sullivan for such weak equivalences. Again, this theory refines the complex homotopy groups of a complex manifold with analytic information. The notion of formality associated to pluripotential weak equivalences turns out to be obstructed by certain Massey-like triple products, called ABC-*Massey products.* These were introduced by Angella and Tomassini in [AT15] for Bott–Chern and Aeppli classes and their study has become an active field of research [ST22], [PSZ24], [ST24]. Interestingly, some compact Kähler manifolds have non-vanishing ABC-Massey products, and are therefore non-formal in the pluripotential sense.

A successful approach to study formality, and more generally, homotopy types of differential graded algebras (dga's) is through Homotopy Transfer Theory: over a field, one can always construct a contraction connecting a dga to its cohomology, which induces an A_∞-structure on cohomology, unique up to A_∞-isomorphism [Kad80]. This A_∞-structure is closely related to Massey products and encodes the homotopy type of the dga. When applied to the differential forms of a smooth manifold or, more generally, to the piece-wise linear forms of a topological space, the A_∞-structure is in fact a C_∞-structure and encodes the rational homotopy type of the space.

As an A_∞-counterpart of the homotopy theory of filtered algebras developed by Neisendorfer–Taylor and Halperin–Tanré, in [CSG22], we prove a Homotopy Transfer Theorem for filtered algebras, endowing any page of the associated spectral sequence with A_∞- and C_∞-structures. This addresses the middle row of the diagram of cohomology theories from a homotopy transfer perspective. In summary, we have the following table:

	Free minimal models	Homotopy transfer
H_{dR}	Sullivan	Kadeishvili
$H_\partial \Rightarrow H_{\mathrm{dR}} \Leftarrow H_{\bar\partial}$	Neisendorfer & Taylor Halperin & Tanré	Cirici & Sopena-Gilboy
$H_{\mathcal{A}}$ & $H_{\mathcal{BC}}$	Stelzig	?

This thesis replaces the above question mark with the present 126 pages. Nowadays, operad theory provides the appropriate framework for homotopy transfer. Here, we develop a new operadic framework in the symmetric monoidal category of bicomplexes with pluripotential weak equivalences.

❧ ❧ ❧

In the first chapter, we study the ground category of the thesis: the category $\mathrm{BiCo}_{\mathbf{k}}$ of bicomplexes over a field $\mathbf{k}$ of characteristic zero. This category admits a model structure with pluripotential weak equivalences [Ste25a]. From the model structure we obtain the notion of *pluripotential homotopy* between two morphisms of bicomplexes f and g given by the equation

$$[\partial, [\overline{\partial}, h]] = f - g.$$

This identity resembles the pluripotential equations $\omega = \partial\bar{\partial}f$ studied as a complex counterpart of classical potential equations $\omega = df$. Note that the homotopy in this context is a map of bidegree (-1, -1).

Cochain complexes split (non-canonically) into indecomposable complexes of the form $\mathbf{k}$ or $\mathbf{k} \xrightarrow{\cong} \mathbf{k}$. Likewise, for any bicomplex $(A, \partial, \bar{\partial})$, we have a decomposition

$$A = A_{\mathrm{sq}} \oplus A_{\mathrm{zig}}$$

where A_{sq} is a direct sum of squares, of the form

$$\begin{tikzcd} \mathbf{k} \arrow[r, "\partial"] & \mathbf{k} \\ \mathbf{k} \arrow[u, "\bar\partial"] \arrow[r, "\partial"'] & \mathbf{k} \arrow[u, "\bar\partial"'] \end{tikzcd},$$

and A_{zig} is a direct sum of bicomplexes of the form

$$\mathbf{k}\ , \quad \begin{tikzcd} \mathbf{k} \\ \mathbf{k} \arrow[u, "\bar\partial"] \end{tikzcd}, \quad \mathbf{k} \xrightarrow{\partial} \mathbf{k}\ , \quad \begin{tikzcd} \mathbf{k} & \\ \mathbf{k} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathbf{k} \end{tikzcd}, \quad \begin{tikzcd} \mathbf{k} & \\ \mathbf{k} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathbf{k} \\ & \mathbf{k} \arrow[u, "\bar\partial"] \end{tikzcd} \quad \cdots$$

In squares and zig-zags, ∂ and $\bar\partial$ are isomorphisms. Note that the composition $\partial\bar\partial$ gives zero on zig-zags. An easy exercise shows that Bott-Chern and Aeppli cohomology vanish on squares but do not vanish, simultaneously, on zig-zags. This translates to the fact that squares do not contain interesting homotopical information, whereas zig-zags do. This observation motivates the definition of a *minimal bicomplex*: a bicomplex satisfying $\partial\bar\partial = 0$.

As a first result, we establish the existence of a *contraction*

$$\begin{tikzcd} (A, \partial, \bar\partial) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (H, \partial, \bar\partial) \arrow[l, shift left, "g"] \end{tikzcd}\ ; \ \text{ with } \partial\bar\partial = 0 \text{ on } H,$$

where f and g are morphisms of bicomplexes such that $fg = \mathrm{id}_H$, and h is a pluripotential homotopy from the identity to gf

$$[\partial, [\overline{\partial}, h]] = \mathrm{id}_A - gf.$$

It turns out that $H \cong H_{\mathcal{BC}}(A) \oplus \tilde{H}_{\mathcal{A}}(A)$, where $\tilde{H}_{\mathcal{A}}(A)$ is the cokernel of the map $H_{\mathcal{BC}}(A) \to H_{\mathcal{A}}(A)$ from Bott-Chern to Aeppli cohomology. Morally, this result implies that, under a choice of sections, any bicomplex admits its 'cohomology' as a deformation retract.

The notion of homotopy introduced above is governed by a shift functor

$$\llcorner : \mathrm{BiCo}_{\mathbf{k}} \longrightarrow \mathrm{BiCo}_{\mathbf{k}}$$

given by $\llcorner(A) = \llcorner \otimes A$, where

$$\llcorner := \begin{tikzcd} \mathbf{k} & \\ \mathbf{k} \arrow[u, "1"] \arrow[r, "-1"'] & \mathbf{k} \end{tikzcd}$$

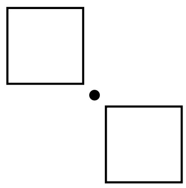
is a bicomplex with the bottom left corner sitting in bidegree (-1, -1). This functor has a homotopy inverse

$$\urcorner : \mathrm{BiCo}_{\mathbf{k}} \longrightarrow \mathrm{BiCo}_{\mathbf{k}}$$

with

$$\urcorner := \begin{tikzcd} \mathbf{k} \arrow[r, "-1"] & \mathbf{k} \\ & \mathbf{k} \arrow[u, "1"'] \end{tikzcd}$$

where the top right corner sits in bidegree $(1,1)$. It is easy to see that they are homotopy inverses noting that

$$\llcorner \otimes \urcorner \cong$$  $$\simeq \mathbf{k}.$$

Therefore, suspending and desuspending in this setting adds meaningless information from the homotopical point of view. Moreover, the composition of functors

$$(\llcorner)^n : \mathrm{BiCo}_{\mathbf{k}} \to \mathrm{BiCo}_{\mathbf{k}} \quad \text{is} \quad (\llcorner)^n(A) = \llcorner^{\otimes n} \otimes A,$$

but $\llcorner^{\otimes n}$ is again not a minimal bicomplex, and it adds unnecessary information (many squares). In this thesis, we define a family of bicomplexes $\{\llcorner_n\}_{n\in\mathbb{Z}}$ that generalize the shift functor in a minimal way. These bicomplexes are defined so that

$$\llcorner_n \simeq (\urcorner)^{\otimes n} \text{ and } \llcorner_{\text{-}n} \simeq (\llcorner)^{\otimes n}.$$

These are given, for lower values of n, by

$$\llcorner_2 = \begin{tikzcd} \mathbf{k} \arrow[r, "\cdot 2"] & \mathbf{k} & \\ & \mathbf{k} \arrow[u, "\cdot(-1)"] \arrow[r, "\cdot 1"] & \mathbf{k} \\ & & \mathbf{k} \arrow[u, "\cdot(-2)"] \end{tikzcd}, \qquad \llcorner_{-2} = \begin{tikzcd} \mathbf{k} & & \\ \mathbf{k} \arrow[u, "\cdot 2"] \arrow[r, "\cdot(-1)"] & \mathbf{k} & \\ & \mathbf{k} \arrow[u, "\cdot 1"] \arrow[r, "\cdot(-2)"] & \mathbf{k} \end{tikzcd},$$

$$\llcorner_{-3} = \begin{tikzcd} \mathbf{k} & & & \\ \mathbf{k} \arrow[u, "\cdot 3"] \arrow[r, "\cdot(-1)"'] & \mathbf{k} & & \\ & \mathbf{k} \arrow[u, "\cdot 2"] \arrow[r, "\cdot(-2)"'] & \mathbf{k} & \\ & & \mathbf{k} \arrow[u, "\cdot 1"] \arrow[r, "\cdot(-3)"'] & \mathbf{k} \end{tikzcd}.$$

We use this family to construct the *inflation functor*

$$\mathrm{Inf} : \mathrm{Ch}_{\mathbf{k}} \longrightarrow \mathrm{BiCo}_{\mathbf{k}}$$

from cochain complexes to bicomplexes. For a cochain complex (C, d), it is defined by

$$\operatorname{Inf}(C) = \left(\bigoplus_{n \in \mathbb{Z}} \mathsf{L}_n \otimes C^n, \partial + \tilde{d}, \bar{\partial} + \tilde{d}' \right).$$

The differentials ∂ and $\bar{\partial}$ correspond to the differentials of the bicomplexes L_n and $\tilde{d}$ and $\tilde{d}'$ are differentials connecting $\mathsf{L}_n \otimes C^n$ and $\mathsf{L}_{n+1} \otimes C^{n+1}$, given by the differential of the cochain complex (C, d).

We prove this functor admits a right adjoint, which we identify with a well-known sheaf-theoretic construction in complex geometry. This construction was originally introduced by Bigolin [Big69] as a functor from complex manifolds to complexes of locally free sheaves. It has the property that Bott–Chern and Aeppli cohomologies arise as cohomologies, in certain degrees, of this complex. The same construction gives a functor $\mathcal{B}$ from bicomplexes to cochain complexes with similar properties. Hodge-theoretic aspects of this functor have recently been developed in [Ste25b] and [Pio25].

There are other possible embeddings of the category of cochain complexes into bicomplexes, but the inflation functor is particularly well behaved from the viewpoint of pluripotential homotopy theory. Indeed, it sends quasi-isomorphisms to pluripotential weak equivalences, and it is conservative. Not only that, but it allows us to place the functor $\mathcal{B}$ within a Quillen adjunction, thereby giving its definition a natural homotopical meaning. We prove

Theorem A. *There is a Quillen adjunction*

$$\operatorname{Inf} : \mathrm{Ch}_{\mathbf{k}} \rightleftarrows \mathrm{BiCo}_{\mathbf{k}} : \mathcal{B}\,.$$

Here, $\mathrm{Ch}_{\mathbf{k}}$ is endowed with the projective model structure and $\mathrm{BiCo}_{\mathbf{k}}$ with the pluripotential model structure of [Ste25a].

Next, we study the monoidal properties of the inflation functor when restricted to non-positively graded cochain complexes. Concretely, we obtain

Theorem B. *The functor*

$$\operatorname{Inf} \colon \mathrm{Ch}_{\mathbf{k}}^{\leq 0} \to \mathrm{BiCo}_{\mathbf{k}}^{\urcorner}$$

is colax symmetric monoidal. Moreover, for all C and D in $\mathrm{Ch}_{\mathbf{k}}^{\leq 0}$, the structure morphism $\phi_{C,D}$ is a pluripotential weak equivalence.

Here, $\mathrm{BiCo}_{\mathbf{k}}^{\urcorner}$ denotes the category of bicomplexes restricted to the third quadrant, that is, $A^{p,q} = 0$ for $p > 0$ or $q > 0$. As a direct consequence of the above theorem, one obtains that the functor $\mathcal{B}$ restricted to third-quadrant bicomplexes is lax symmetric monoidal. A less immediate step is to compute the explicit monoidal structure, which we also provide.

⋆ ⋆ ⋆

In Chapter 2, we develop pluripotential operadic theory. Our first goal is to define homotopy algebras in the pluripotential setting. To achieve this, we compute minimal models of operads with respect to the appropriate notion of weak equivalence, namely, pluripotential weak equivalences. We do this in two complementary ways. First, following the more classical approach of Markl, Shnider, and Stasheff [MSS02], we obtain minimal models for any reduced operad over $\mathrm{BiCo}_{\mathbf{k}}$ adding free extensions inductively. In our setting, the notion of minimal operad is characterized by cofibrancy together with the condition that elements in the image of $\partial\bar{\partial}$ are decomposable, that is, a composition of two or more operations. We prove

Theorem C. *Let $\mathcal{P}$ be an operad such that $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \mathbf{k}$. Then, there exists a minimal operad $\mathcal{M}$ and a pluripotential weak equivalence $\mathcal{M} \xrightarrow{\sim} \mathcal{P}$. The operad $\mathcal{M}$ is unique up to isomorphism.*

The second way is through Koszul duality. This takes advantage of the dual category to operads, that is, cooperads. Koszul operads are operads satisfying the necessary conditions so that one can compute their minimal model from its Koszul dual cooperad $\mathcal{P}^{¡}$. The Koszul criterion tells us that a dg-operad is Koszul if its Koszul complex is acyclic. It is clear, thus, that Koszulity is a property that depends on the chosen class of weak equivalences, which in the dg case are quasi-isomorphisms. In the pluripotential case, we consider the analogous definition with the corresponding weak equivalences.

For a quadratic operad $\mathcal{P}$ in $\mathbf{k}$-vector spaces, we define its *pluripotential Koszul dual cooperad* as

$$\mathcal{P}^{¡}_{pp} := \mathrm{Inf}(\mathcal{P}^{¡}),$$

where $\mathcal{P}^{¡}$ denotes the classical Koszul dual cooperad of $\mathcal{P}$, viewed as a dg operad, and Inf denotes the inflation functor introduced in the first chapter. The cooperad structure is given by the cooperad structure of $\mathcal{P}^{¡}$ and the colax structure of Inf.

The pluripotential Koszul dual cooperad allows us to construct the Koszul bicomplex $\mathcal{P} \circ_{\kappa} \mathcal{P}^{¡}_{pp}$ of the operad $\mathcal{P}$, which plays the role of the Koszul complex of a dg operad. We prove that, for any quadratic operad in vector spaces that is Koszul as a dg-operad, there is a pluripotential weak equivalence

$$\mathcal{P} \circ_{\kappa} \mathcal{P}^{¡}_{pp} \xrightarrow{\sim} \mathrm{I}.$$

In particular, Koszulity and pluripotential Koszulity coincide for operads in $\mathbf{k}$-vector spaces. Consequently, the operads *As*, *Com*, and *Lie*, encoding associative, commutative, and Lie algebras, respectively, are Koszul in the pluripotential sense.

We define

$$\mathcal{P}^{pp}_{\infty} := \left(\mathbb{T}(\mathrm{Inf}(s\overline{\mathcal{P}^{¡}})), \partial, \bar{\partial}\right)$$

as the free operad generated by $\mathrm{Inf}(s\mathcal{P}^{¡})$ with two additional differentials. Here, $s\mathcal{P}^{¡}$ denotes the suspension of the Koszul dual cooperad of $\mathcal{P}$ viewed as a dg-operad. We show

Theorem D. *Let $\mathcal{P}$ be a Koszul operad in $\mathbf{k}$-vector spaces. The operad $\mathcal{P}^{pp}_{\infty}$ is the pluripotential minimal model of $\mathcal{P}$.*

From this, we obtain almost for free the pluripotential minimal models of the operads *As*, *Com*, and *Lie*, and hence the definition of pluripotential A_∞, C_∞, and L_∞-algebras. A pluripotential version of ∞-morphisms is obtained through the bar construction as follows.

Let now $\mathcal{P}$ be an operad and $\mathcal{C}$ a cooperad, both in bicomplexes. Given a twisting morphism $\varphi : \mathcal{P} \to \mathcal{C}$, the bar and cobar construction give rise to a pair of adjoint functors

$$\Omega_{\varphi} : \mathcal{C}\text{-coalgebras} \rightleftarrows \mathcal{P}\text{-algebras} : \mathrm{B}_{\varphi}$$

between the category of $\mathcal{C}$-coalgebras and the category of $\mathcal{P}$-algebras. In the classical setting, a twisting morphism is an equivariant map of degree 1 satisfying the Maurer–Cartan equation. Since $\mathcal{P}$ and $\mathcal{C}$ are (co)operads in $\mathrm{BiCo}_{\mathbf{k}}$, the degree condition is replaced by the existence of two components φ^{10} and φ^{01} of bidegrees $(1,0)$ and $(0,1)$. This refines the classical notion: after totalization, the map $\varphi = \varphi^{10} + \varphi^{01}$ becomes a classical twisting morphism with respect to the differential $d = \partial + \bar{\partial}$.

The bar construction provides an alternative description of $\mathcal{P}_\infty^{pp}$-algebras in terms of two anticommuting codifferentials on cofree coalgebras. A $\overset{pp}{\infty}$-morphism between two $\mathcal{P}_\infty^{pp}$-algebras is then defined as a morphism between their bar constructions with respect to a twisting morphism $\iota \colon \mathcal{P}_{pp}^{\text{!`}} \to \mathcal{P}_\infty^{pp}$. We denote by $\infty\text{-}\mathcal{P}_\infty^{pp}\text{-Alg}$ the category of homotopy $\mathcal{P}$-algebras ($\mathcal{P}_\infty^{pp}$-algebras) with $\overset{pp}{\infty}$-morphisms. We have

Theorem E. *Let $\mathcal{P}$ be a Koszul operad in $\mathbf{k}$-vector spaces. There is an equivalence of homotopy categories*

$$\mathrm{Ho}(\mathcal{P}\text{-Alg}) \cong \mathrm{Ho}(\infty\text{-}\mathcal{P}_\infty^{pp}\text{-Alg}).$$

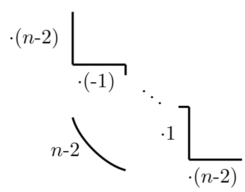
Here, the localization Ho *is with respect to pluripotential weak equivalences.*

As in the dg case, $\mathcal{P}_\infty^{pp}$-algebras do not yield 'new homotopy types' of algebras but a new description of the existing ones.

In the bicomplex context, algebras over the associative operad are *bidifferential bigraded algebras*, and algebras over its pluripotential minimal model

$$A_\infty^{pp} = \left(\mathbb{T}(\mathrm{Inf}(sAs^{\text{!`}})), \partial, \bar{\partial}\right)$$

are *pluripotential A_∞-algebras* (A_∞^{pp}-algebras for short). Note that the n-ary generating operations are organized in the bicomplex

$\llcorner_{2\text{-}n} :=$ 

In particular, in arity n, we have $n-2$ generating operations of total degree $1-n$ and $n-1$ generating operations of total degree $2-n$.

As expected, A_∞^{pp}-algebras are algebraic structures in which associativity holds up to higher homotopies. Explicitly, a A_∞^{pp}-algebra is a bicomplex $(A, \partial, \bar{\partial})$ together with n-ary operations

$$\mu_n^{p,q} : A^{\otimes n} \longrightarrow A$$

for $n \geq 2$ and bidegree (p,q) with $1-n \leq p+q \leq 2-n$ satisfying certain associativity relations. For instance, there is a single binary operation μ_2 of bidegree zero satisfying

$$\mu_2(\mu_2 \otimes \mathrm{id}) - \mu_2(\mathrm{id} \otimes \mu_2) = [\partial, [\overline{\partial}, \mu_3]].$$

Here, μ_3 is a ternary operation of bidegree (-1,-1). Note that this equation states that μ_3 is a pluripotential homotopy measuring the failure of μ_2 to be associative. We prove the following

Homotopy Transfer Theorem.

Theorem F. *Let $(A, \partial, \bar{\partial})$ be a bidifferential bigraded algebra.*

1. *There is a contraction into a minimal bicomplex*

$$\begin{tikzcd} (A,\partial,\bar{\partial}) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (H,\partial,\bar{\partial}) \arrow[l, shift left, "g"] \end{tikzcd} \; ; \quad \textit{with } \partial\bar{\partial} = 0 \textit{ on } H.$$

2. *There is a A_∞^{pp}-structure on $(H, \partial, \bar{\partial})$ and a $\overset{pp}{\infty}$-weak equivalence $g : H \rightsquigarrow A$.*

As mentioned before, $H \cong H_{\mathcal{BC}}(A) \oplus \tilde{H}_{\mathcal{A}}(A)$, where $\tilde{H}_{\mathcal{A}}(A)$ is the cokernel of the map $H_{\mathcal{BC}}(A) \to H_{\mathcal{A}}(A)$. Thus, this theorem provides a way of encoding the full pluripotential homotopy type of the initial bidifferential bigraded algebra into its cohomology. When the algebra satisfies the $\partial\bar{\partial}$-property, H can be chosen to be the Bott-Chern cohomology $H_{\mathcal{BC}}$, and the binary product μ_2 coincides with the induced product on cohomology. In fact, when the original algebra is commutative, we obtain a C_∞^{pp}-structure on H.

We also develop the pluripotential counterpart of the theory of cyclic A_∞-algebras (see [Kon93], [Kon94]). Our cyclic version of the homotopy transfer theory is motivated by the Poincaré-Serre duality theory of complex manifolds.

⋆ ⋆ ⋆

In Chapter 3, we develop two main applications for complex manifolds. The first one is the explicit link between the ABC-Massey products of [AT15] and our transferred structures: we prove that transferred C_∞^{pp}-structures recover these products in arity three. This interpretation places them in a homotopical framework where higher operations arise naturally.

The second application deals with the two natural notions of formality associated with pluripotential weak equivalences, that is, *weak* and *strong formality* defined in [MS24] and further studied in [Ste25a], [PSZ25], [ST25]. A *weakly formal* complex manifold is a manifold whose complexified de Rham algebra is connected by a zig-zag of pluripotential weak equivalences to a cbba satisfying $\partial\bar{\partial} = 0$, if it satisfies the stronger condition $\partial = \bar{\partial} = 0$, then the manifold is *strongly formal.* When the transferred C_∞-algebra H satisfies that the operations of arity $n \geq 3$ vanish, then the algebra is weakly formal. If the algebra satisfies the $\partial\bar{\partial}$-property, then it is strongly formal.

On a compact complex manifold M, there is a pairing

$$\langle\, , \rangle : \mathcal{A}_{\mathrm{dR}}(M; \mathbb{C}) \times \mathcal{A}_{\mathrm{dR}}(M; \mathbb{C}) \to \mathbb{C}$$

given by integration of forms

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta.$$

This encodes Poincaré duality. Abstractly, the interaction between algebraic structures and pairings is well described in terms of cyclic algebras. The algebra $\mathcal{A}_{\mathrm{dR}}(M; \mathbb{C})$ is a cyclic cbba in the sense that

$$\langle \alpha \wedge \beta, \gamma \rangle = (-1)^{|\gamma|(|\alpha|+|\beta|)} \langle \gamma \wedge \alpha, \beta \rangle.$$

In particular, the composition of the pairing with the product gives a map

$$\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})^{\otimes 3} \to \mathbb{C}$$

which is invariant to cyclic permutations of the entries. In the Kähler case, we make use of Hodge theory to define a contraction

$$\begin{tikzcd} \mathcal{A}_{\mathrm{dR}}(M;\mathbb{C}) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & \mathcal{H} \arrow[l, shift left, "g"] \end{tikzcd}$$

compatible with the pairing mentioned above. Here, $\mathcal{H}$ denotes the space of harmonic forms. We use this contraction to equip $\mathcal{H}$ with a cyclic C_∞^{pp}-algebra. It is worth mentioning that the wedge product of harmonic forms is not harmonic in general. However, this procedure equips it with multiplicative structure. Moreover, the binary operation obtained is an associative product, since the differentials are trivial on $\mathcal{H}$. Taking advantage of cyclicity, we prove

Theorem G (cf. Theorem 3.5 in [Ste25a])**.** *Any compact Kähler manifold M of complex dimension $n \geq 2$ with the Hodge diamond of a complete intersection is strongly formal.*

This theorem was already proven by Stelzig in [Ste25a] using minimal models. However, our approach gives a much more direct and concise proof. We hope to further exploit our methods to prove further formality results in the future.

Chapter 1

Bicomplexes

In this chapter, we study the base category of this thesis: the category of bicomplexes $\mathrm{BiCo}_{\mathbf{k}}$ over a field of characteristic zero. We recall the notion of pluripotential weak equivalence and related structure properties. We show that any bicomplex admits a contraction into a minimal bicomplex in the pluripotential sense.

Furthermore, we introduce a functor from cochain complexes to bicomplexes, called the *inflation functor*, which sends quasi-isomorphisms to pluripotential weak equivalences. We prove that this functor is part of a Quillen adjunction. Its right adjoint is a classical construction in complex geometry, which gives a sheaf-theoretic presentation of Bott-Chern and Aeppli cohomologies.

When restricted to non-positive degrees, we show that the inflation functor is colax symmetric monoidal. Consequently, its right adjoint is lax symmetric monoidal. These properties will play a central role in Chapter 2, for the development of pluripotential Koszul duality theory for operads.

Throughout this chapter, $\mathbf{k}$ denotes a field of characteristic zero. Although most constructions can be carried out over more general rings, this assumption is sufficient for our purposes.

1.1 The category of bicomplexes

Denote by $\mathrm{BiCo}_{\mathbf{k}}$ the category of bicomplexes over $\mathbf{k}$. Its objects are given by bigraded vector spaces

$$A = \bigoplus_{p,q\in\mathbb{Z}} A^{p,q}$$

together with two linear maps ∂ and $\bar{\partial}$ of bidegrees $(1,0)$ and $(0,1)$, respectively, and satisfying the relations

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0 \quad \text{and} \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Morphisms in this category are given by bidegree-preserving linear maps compatible with both ∂ and $\bar{\partial}$.

Remark 1.1.1. Although we use the notation ∂ and $\bar{\partial}$ for the differentials of a bicomplex, borrowed from the notation of complex manifolds, our bicomplexes are defined over an arbitrary field, so we do not assume any real structure.

The category $\mathrm{BiCo}_{\mathbf{k}}$ is a symmetric monoidal category, with the tensor product given by

$$(A \otimes B)^{p,q} = \bigoplus_{r,s\in\mathbb{Z}} A^{r,s} \otimes B^{p-r,q-s}$$

with differentials given by the Leibniz rule. The unit is given by the field $\mathbf{k}$ concentrated in bidegree $(0,0)$.

Given bigraded linear maps f and g, denote by $[f,g]$ the graded commutator

$$[f,g] := f \circ g - (-1)^{|f|\cdot|g|} g \circ f,$$

where $|f|$ denotes the total degree of f. The internal Hom is defined by

$$\underline{\mathrm{Hom}}(A,B)^{p,q} = \prod_{r,s\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{k}}(A^{r,s}, B^{r+p,s+q}),$$

with differentials

$$\partial f := [\partial, f] \quad \text{and} \quad \bar{\partial} f := [\bar{\partial}, f].$$

The internal Hom and tensor product form an adjunction

$$(-\otimes A) : \mathrm{BiCo}_{\mathbf{k}} \rightleftarrows \mathrm{BiCo}_{\mathbf{k}} : \underline{\mathrm{Hom}}(A,-).$$

The category $\mathrm{BiCo}_{\mathbf{k}}$ is then a closed symmetric monoidal category.

Associated to a bicomplex $(A, \partial, \bar{\partial})$ there are various functorial cohomologies: aside from the ∂- and $\bar{\partial}$-cohomologies

$$H^{*,*}_{\partial}(A) := \frac{\mathrm{Ker}(\partial)}{\mathrm{Im}(\partial)} \quad \text{and} \quad H^{*,*}_{\bar{\partial}}(A) := \frac{\mathrm{Ker}(\bar{\partial})}{\mathrm{Im}(\bar{\partial})},$$

we also have the Bott-Chern $H_{\mathcal{BC}}$ and Aeppli $H_{\mathcal{A}}$ cohomologies, given respectively by

$$H^{*,*}_{\mathcal{BC}}(A) := \frac{\mathrm{Ker}(\bar{\partial}) \cap \mathrm{Ker}(\partial)}{\mathrm{Im}(\bar{\partial}\partial)} \quad \text{and} \quad H^{*,*}_{\mathcal{A}}(A) := \frac{\mathrm{Ker}(\bar{\partial}\partial)}{\mathrm{Im}(\bar{\partial}) + \mathrm{Im}(\partial)}.$$

Note that the identity induces natural transformations

$$\begin{tikzcd} & H_{\mathcal{BC}} \arrow[dl] \arrow[d] \arrow[dr] & \\ H_{\partial} \arrow[dr] & H_{dR} \arrow[d] & H_{\bar{\partial}} \arrow[dl] \\ & H_{\mathcal{A}} & \end{tikzcd}$$

where H_{dR} denotes the total cohomology. Moreover, the row and column filtrations on a bicomplex A induce two spectral sequences relating the total cohomology with ∂- and $\bar{\partial}$- cohomologies, respectively:

$$H_{\partial}(A) \Rightarrow H_{dR}(A) \Leftarrow H_{\bar{\partial}}(A).$$

Definition 1.1.2. A bicomplex A is said to *satisfy the $\partial\bar{\partial}$-property* if the map of bigraded vector spaces

$$H_{\mathcal{BC}}(A) \to H_{\mathcal{A}}(A),$$

is injective.

Remark 1.1.3. In fact, if the map $H_{\mathcal{BC}}(A) \to H_{\mathcal{A}}(A)$ is injective, then it is an isomorphism, and all the maps in the above diagram are also isomorphisms (see [DGMS75]).

Remark 1.1.4. If a bicomplex $(A, \partial, \bar{\partial})$ satisfies the $\partial\bar{\partial}$-property and $a \in A$ verifies $\partial a = \bar{\partial} a = 0$, then

1. If $\alpha = \partial\gamma$ for some γ, then there exists β such that $\alpha = \partial\bar{\partial}\beta$.
2. If $\alpha = \bar{\partial}\gamma$ for some γ, then there exists β such that $\alpha = \partial\bar{\partial}\beta$.

Consider the natural transformation induced by the identity $H_{\mathcal{BC}} \to H_{\mathcal{A}}$, and let $\widetilde{H}_{\mathcal{BC}}$ and $\widetilde{H}_{\mathcal{A}}$ be the functors which send a bicomplex A to the kernel and cokernel of $H_{\mathcal{BC}}(A) \to H_{\mathcal{A}}(A)$, respectively. Explicitly, this gives:

$$\widetilde{H}_{\mathcal{BC}} = \frac{\mathrm{Im}\partial \cap \mathrm{Ker}\bar{\partial} + \mathrm{Im}\bar{\partial} \cap \mathrm{Ker}\partial}{\mathrm{Im}\partial\bar{\partial}} \quad \text{and} \quad \widetilde{H}_{\mathcal{A}} = \frac{\mathrm{Ker}\partial\bar{\partial}}{\mathrm{Im}\partial + \mathrm{Im}\bar{\partial} + \mathrm{Ker}\partial \cap \mathrm{Ker}\bar{\partial}}.$$

The *ABC-cohomology* of a bicomplex $(A, \partial, \bar{\partial})$ is the bicomplex defined by

$$H_{\mathcal{ABC}}(A) := H_{\mathcal{BC}}(A) \oplus \widetilde{H}_{\mathcal{A}}(A)$$

together with the two differentials ∂ and $\bar{\partial}$ induced from those of A. Note that this bicomplex is *minimal*, in the sense that $\partial\bar{\partial} \equiv 0$.

Lemma 1.1.5 ([Ste25a, Cor. 1.12])**.** *Any short exact sequence of bicomplexes*

$$0 \to A \to B \to C \to 0$$

induces a long exact sequence

$$\begin{aligned} \cdots \to H^{p+q-2}_{\mathcal{B}_{p,q}}(B) \to H^{p+q-2}_{\mathcal{B}_{p,q}}(C) \to H^{p-1,q-1}_{\mathcal{A}}(A) \to H^{p-1,q-1}_{\mathcal{A}}(B) \to H^{p-1,q-1}_{\mathcal{A}}(C) \to \\ \to H^{p,q}_{\mathcal{BC}}(A) \to H^{p,q}_{\mathcal{BC}}(B) \to H^{p,q}_{\mathcal{BC}}(C) \to H^{p+q+1}_{\mathcal{B}_{p,q}}(A) \to H^{p+q+1}_{\mathcal{B}_{p,q}}(B) \to \dots \end{aligned}$$

where $H^k_{\mathcal{B}_{p,q}}$ denotes the cohomology of the Bigolin complex in degree k, see Section 1.3.

Any bicomplex A admits a (non-canonical) decomposition

$$A = A_{\mathrm{zig}} \oplus A_{\mathrm{sq}}$$

where A_{sq} is a direct sum of squares, of the form

$$\begin{tikzcd} \mathbf{k} \arrow[r, "\partial"] & \mathbf{k} \\ \mathbf{k} \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"'] & \mathbf{k} \arrow[u, "\bar{\partial}"'] \end{tikzcd},$$

and A_{zig} is a bicomplex satisfying $\partial\bar{\partial} = 0$. Therefore, A_{zig} is a direct sum of bicomplexes of the form

$$\mathbf{k}\ ,\qquad \begin{tikzcd} \mathbf{k} \\ \mathbf{k} \arrow[u, "\bar{\partial}"] \end{tikzcd}\ ,\qquad \begin{tikzcd} \mathbf{k} \arrow[r, "\partial"] & \mathbf{k} \end{tikzcd}\ ,\qquad \begin{tikzcd} \mathbf{k} & \\ \mathbf{k} \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"] & \mathbf{k} \end{tikzcd}\ ,\qquad \begin{tikzcd} \mathbf{k} & \\ \mathbf{k} \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"] & \mathbf{k} \\ & \mathbf{k} \arrow[u, "\bar{\partial}"] \end{tikzcd}\quad \cdots$$

In squares and zig-zags, ∂ and $\bar{\partial}$ are isomorphisms. This result has long been known as a folklore statement, complete proofs can be found in [Ste21] and [KQ20].

With such a decomposition, we have $A_{\text{zig}} \cong H_{\mathcal{ABC}}(A)$. In particular, the summand A_{sq} is acyclic in terms of Bott-Chern and Aeppli cohomologies. If the bicomplex satisfies the $\partial\bar{\partial}$-lemma property, the bicomplex A_{zig} has trivial differentials, and hence the bicomplex A splits as a direct sum of squares and a bigraded vector space.

Definition 1.1.6. A bicomplex A is called *bounded* if $A^{p,q} = 0$ for all but finitely many $(p, q) \in \mathbb{Z} \times \mathbb{Z}$.

The category of bicomplexes admits a model structure whose weak equivalences are defined in terms of Bott–Chern and Aeppli cohomologies. The rest of this section reviews this model structure and the relevant definitions.

Definition 1.1.7. A morphism of bicomplexes $f\colon A \to B$ is said to be a *pluripotential weak equivalence* if $H_{\mathcal{A}}(f)$ and $H_{\mathcal{BC}}(f)$ are isomorphisms.

Remark 1.1.8. Being a pluripotential weak equivalence $f\colon A \to B$ is equivalent to inducing an isomorphism $H_{\mathcal{ABC}}(f)$ at the level of ABC-cohomology. In [Ste25a], pluripotential weak equivalences are also called *pluripotential quasi-isomorphism* or a *bigraded quasi-isomorphism*.

Remark 1.1.9. If $f\colon A \to B$ is a map between bounded bicomplexes, then f is a pluripotential weak equivalence if and only if it induces an isomorphism in cohomology with respect to ∂ and with respect to $\bar{\partial}$. See Theorem 1.21 in [Ste25a].

The category $\mathrm{BiCo}_{\mathbf{k}}$ admits a cofibrantly generated model structure (see [Ste25a, Theorem 1.7]), where:

- Fibrations are surjective morphisms,
- weak equivalences are pluripotential weak equivalences,
- cofibrations are injective morphisms.

We will refer to this as the *pluripotential model structure*. It is easy to see that every object is fibrant and cofibrant. The tensor product and internal hom make $\mathrm{BiCo}_{\mathbf{k}}$ into a symmetric monoidal model category. The notion of homotopy associated with this model structure is the following.

Definition 1.1.10. Let $f, g\colon A \to B$ be two maps of bicomplexes. A *pluripotential homotopy from f to g* is a linear map $h\colon A \to B$ of bidegree $(-1,-1)$ satisfying

$$[\partial, [\bar{\partial}, h]] = f - g.$$

If such a homotopy exists, we write $f \simeq g$.

Since we are working over a field, the class of homotopy equivalences induced from the above definition coincides with the class of pluripotential weak equivalences (see Theorem 1.21 [Ste25a]).

Proposition 1.1.11. *Any two homotopic maps $f, g\colon A \to B$ of bicomplexes induce the same map in cohomology:*

$$[f]_{\mathcal{BC}} = [g]_{\mathcal{BC}} : H_{\mathcal{BC}}(A) \to H_{\mathcal{BC}}(B) \quad \textit{and} \quad [f]_{\mathcal{A}} = [g]_{\mathcal{A}} : H_{\mathcal{A}}(A) \to H_{\mathcal{A}}(B).$$

We will use the following notion of shift functor $\llcorner : \mathrm{BiCo}_{\mathbf{k}} \to \mathrm{BiCo}_{\mathbf{k}}$ as defined in [Ste25a]. For a bicomplex A, let

$$\llcorner(A) := \llcorner \otimes A,$$

where

$$\llcorner := \begin{tikzcd} \mathbf{k} & \\ \mathbf{k} \arrow[u, "1"] \arrow[r, "-1"'] & \mathbf{k} \end{tikzcd}$$

and the bottom left corner is sitting in bidegree $(-1,-1)$. A homotopy inverse $\urcorner : \mathrm{BiCo}_{\mathbf{k}} \to \mathrm{BiCo}_{\mathbf{k}}$ of $\llcorner$ is given by

$$\urcorner(A) := \urcorner \otimes A,$$

where

$$\urcorner := \begin{tikzcd} \mathbf{k} \arrow[r, "-1"] & \mathbf{k} \\ & \mathbf{k} \arrow[u, "1"'] \end{tikzcd}$$

and the top right corner is sitting in bidegree $(1,1)$.

Remark 1.1.12. Note that

$$\llcorner \otimes \urcorner \cong \begin{tikzcd} \mathbf{k}^{-1,1} \arrow[r] & \mathbf{k} & & & \\ \mathbf{k} \arrow[u] \arrow[r] & \mathbf{k} \arrow[u] & \mathbf{k}^{0,0} & \mathbf{k} \arrow[r] & \mathbf{k} \\ & & & \mathbf{k} \arrow[u] \arrow[r] & \mathbf{k}^{1,-1} \arrow[u] \end{tikzcd} \simeq \mathbf{k},$$

where $\mathbf{k}$ is the monoidal unit, and the superscripts indicate the bidegree of the vector space.

The following definition introduces the pluripotential analogue of the mapping cone for cochain complexes. It will be used to detect pluripotential weak equivalences.

Definition 1.1.13. The *mapping cone* of a morphism of bicomplexes $f\colon A \to B$ is the bicomplex defined by

$$C(f) := \mathsf{L}(A) \oplus B,$$

with differentials given by

$$\begin{aligned}\partial(a, a', a'', b) &= (\partial a, -\partial a', -\partial a'' - a, \partial b + f(a')),\\ \bar\partial(a, a', a'', b) &= (\bar\partial a, -\bar\partial a' + a, -\bar\partial a'', \bar\partial b + f(a'')).\end{aligned}$$

There is a short exact sequence

$$B \to C(f) \to \mathsf{L}A.$$

Using Theorem 1.21 of [Ste25a], we see that a map f is a pluripotential weak equivalence if and only if $C(f)$ is acyclic, that is, $H_{\mathcal{BC}}(C(f)) = 0$.

We next state Künneth formulae for Bott-Chern and Aeppli cohomologies. There are natural maps

$$H_{\mathcal{BC}}(A) \otimes H_{\mathcal{BC}}(B) \longrightarrow H_{\mathcal{BC}}(A \otimes B)$$

and

$$H_{\mathcal{BC}}(A) \otimes H_{\mathcal{A}}(B) \ \oplus\ H_{\mathcal{A}}(A) \otimes H_{\mathcal{BC}}(B) \longrightarrow H_{\mathcal{A}}(A \otimes B).$$

These maps are neither injective nor surjective in general, see Section 1.7 of [Ste25a]. Below, we provide Künneth-type formulas for $H_{\mathcal{ABC}}$ cohomology, although the resulting isomorphisms are not natural. For canonical descriptions of the kernels and cokernels of the maps above, see Theorem 1.35 in [Ste25a].

Proposition 1.1.14. *Let A and B be two bicomplexes. Then, there exists an isomorphism*

$$H_{\mathcal{ABC}}(A \otimes B) \cong H_{\mathcal{ABC}}(H_{\mathcal{ABC}}(A) \otimes H_{\mathcal{ABC}}(B)).$$

Proof. If $A \simeq B$ is a pluripotential homotopy equivalence, then $A \otimes C \simeq B \otimes C$ as well. One may take a homotopy of the form $h \otimes \mathrm{id}_C$, where h is a homotopy witnessing $A \simeq B$. Therefore, by Theorem 1.21 in [Ste25a], for any choice of decomposition into squares and zig-zags, we obtain

$$A \otimes B \simeq A_{\mathrm{zig}} \otimes B_{\mathrm{zig}}.$$

Choosing an isomorphism $H_{\mathcal{ABC}}(A) \cong A_{\mathrm{zig}}$ and $H_{\mathcal{ABC}}(B) \cong B_{zig}$ and using Proposition 1.1.11 we obtain an isomorphism

$$H_{\mathcal{ABC}}(A \otimes B) \cong H_{\mathcal{ABC}}(H_{\mathcal{ABC}}(A) \otimes H_{\mathcal{ABC}}(B)). \qquad \square$$

Remark 1.1.15. The proposition above relies on the fact that any bicomplex A splits (non-uniquely) as a direct sum of squares and zig-zags,

$$A = A_{\mathrm{zig}} \oplus A_{\mathrm{sq}}.$$

The hypotheses required for such a decomposition can be weakened to bicomplexes in other abelian categories, for instance, to bicomplexes of characteristic-zero representations of a finite group G, see Remark 5 in [Ste21]. Consequently, Proposition 1.1.14 also applies to bicomplexes in $\mathbf{k}[\mathbb{S}_n]$-modules, a fact we will use in Chapter 2.

1.2 Contractions

In the following, we will define contractions in the pluripotential setting, a key concept for the homotopy transfer theory developed in Chapter 2, and show that any bicomplex admits a contraction into a minimal bicomplex.

Definition 1.2.1. A *contraction* of a bicomplex $(A, \partial, \bar{\partial})$ into a bicomplex $(B, \partial, \bar{\partial})$ is a diagram

$$\begin{tikzcd} (A, \partial, \bar{\partial}) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (B, \partial, \bar{\partial}) \arrow[l, shift left, "g"] \end{tikzcd}$$

where f and g are bicomplex morphisms such that $fg = \mathrm{id}_B$ and h is a pluripotential homotopy from the gf to the identity:

$$gf - \mathrm{id}_A = [\partial, [\bar{\partial}, h]].$$

If (f, g, h) is a contraction, we can define

$$h_{10} := [\bar{\partial}, h] \quad \text{and} \quad h_{01} := -[\partial, h].$$

These maps satisfy

$$gf - \mathrm{id}_A = [\partial, h_{10}] \quad \text{and} \quad gf - \mathrm{id}_A = [\bar{\partial}, h_{01}].$$

In particular, h_{10} and h_{01} are chain homotopies from the composition gf to the identity id_A, with respect to ∂ and $\bar{\partial}$, respectively.

The following definition will be used in Chapter 2 to obtain homotopy transfer with units.

Definition 1.2.2. A contraction is said to satisfy the *side conditions* if

$$f \circ h = 0, \quad h \circ g = 0, \quad h \circ h = 0, \quad h\partial h = 0, \quad \text{and} \quad h\bar{\partial}h = 0.$$

Remark 1.2.3. The side conditions imply

$$f \circ h_{10} = 0, \quad h_{10} \circ g = 0, \quad f \circ h_{01} = 0, \quad \text{and} \quad h_{01} \circ g = 0,$$

and

$$h \circ h_{10} = h_{10} \circ h = 0, \quad h \circ h_{01} = h_{01} \circ h = 0, \quad \text{and} \quad h_{10}h_{01} = -h_{01}h_{10} = h.$$

Proposition 1.2.4. *Every bicomplex $(A, \partial, \bar{\partial})$ admits a contraction into a minimal bicomplex:*

$$\begin{tikzcd} (A, \partial, \bar{\partial}) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (H, \partial, \bar{\partial}) \arrow[l, shift left, "g"] \end{tikzcd} \; ; \; \textit{with } \partial\bar{\partial} = 0 \textit{ on } H.$$

Proof. By Section 1.1, we can decompose (non-uniquely) every bicomplex A as

$$A = A_{\mathrm{sq}} \oplus A_{\mathrm{zig}}.$$

Let $H := A_{\mathrm{zig}} \simeq A$. We define $f : A \twoheadrightarrow H$ and $g : H \hookrightarrow A$ via the decomposition. Next, we define the homotopy

$$h : A_{\mathrm{sq}} \oplus A_{\mathrm{zig}} \to A_{\mathrm{sq}} \oplus A_{\mathrm{zig}}$$

as follows. Given a pair $(a, b) \in A_{\mathrm{sq}} \oplus A_{\mathrm{zig}}$, we let $h(a, b) = (a', 0)$ if $a = -\partial\bar{\partial}a'$ and zero otherwise. Note that since we are considering a splitting into squares and zig-zags, ∂ and $\bar{\partial}$ are isomorphisms, so a' is uniquely determined. This homotopy satisfies

$$gf - \mathrm{id}_A = [\partial, [\bar{\partial}, h]].$$

First, observe that for any element $(a, b) \in A_{\mathrm{sq}} \oplus A_{\mathrm{zig}}$, we have:

$$(gf - \mathrm{id}_A)(a, b) = -(a, 0).$$

Next, note that

$$[\partial, [\bar{\partial}, h]](A_{\mathrm{zig}}) = 0,$$

by definition. Now, consider the case when $a \in A_{\mathrm{sq}}$, we will compute

$$[\partial, [\bar{\partial}, h]]a = (\partial\bar{\partial}h - \partial h\bar{\partial} + \bar{\partial}h\partial - h\bar{\partial}\partial)a.$$

Assume a is a degree-homogeneous element in a square, as depicted below:

$$\begin{tikzcd}
\bar{\partial}\alpha \arrow[r, "\partial"] & \partial\bar{\partial}\alpha \\
\alpha \arrow[u, "\bar{\partial}"] \arrow[r, "\partial"'] & \partial\alpha \arrow[u, "\bar{\partial}"']
\end{tikzcd}$$

There are four possible cases for a based on its position in the square:

1. If a is the top right corner of a square, so that $a = \partial\bar{\partial}\alpha$, then the only non zero term is

$$\partial\bar{\partial}ha = -\partial\bar{\partial}\alpha = -a.$$

2. If a is the bottom right corner of a square, so that $a = \partial\alpha$, then the only non zero term is

$$-\partial h\bar{\partial}a = -\partial h\bar{\partial}\partial\alpha = -\partial(\alpha) = -a.$$

3. If a is the top left corner of a square, so that $a = \bar{\partial}\alpha$, then

$$\bar{\partial}h\partial a = \bar{\partial}h\partial\bar{\partial}\alpha = -\bar{\partial}\alpha = -a$$

is the only non-zero term.

4. If a is the bottom left corner of a square, then

$$-h\bar{\partial}\partial a = -\alpha = -a$$

is the only non-zero term.

In all cases, we obtain $[\partial, [\bar{\partial}, h]]a = -a$. Therefore, we conclude

$$[\partial, [\bar{\partial}, h]](a, b) = -(a, 0),$$

which completes the proof. $\square$

Remark 1.2.5. Any contraction constructed as in Proposition 1.2.4 satisfies the side conditions.

1.3 The Bigolin complex

The *Bigolin complex* was introduced by Bigolin [Big69] and further studied by Schweitzer [Sch07] and Stelzig in [Ste25b], who calls it the Schweitzer complex. We recall its definition and review some of its properties. We will denote by $\mathrm{Ch}_{\mathbf{k}}$ the category of cochain complexes over $\mathbf{k}$.

Definition 1.3.1. Let $(A, \partial, \bar{\partial})$ be a bicomplex. Define the cochain complex $(\mathcal{B}_{p,q}(A), d_{\mathcal{B}})$ as follows: as a graded vector space, we let

$$\mathcal{B}^k_{p,q}(A) := \bigoplus_{\substack{r+s=k-1\\ r<p, s<q}} A^{r,s} \qquad \text{if } k \leq p+q-1$$
$$\mathcal{B}^k_{p,q}(A) := \bigoplus_{\substack{r+s=k\\ r\geq p, s\geq q}} A^{r,s} \qquad \text{if } k \geq p+q$$

The differential $d_{\mathcal{B}}$ is given by:

$$\cdots \xrightarrow{\mathrm{pr}d} \mathcal{B}^{p+q-2}_{p,q}(A) \xrightarrow{\mathrm{pr}d} \mathcal{B}^{p+q-1}_{p,q}(A) \xrightarrow{\partial\bar{\partial}} \mathcal{B}^{p+q}_{p,q}(A) \xrightarrow{d} \mathcal{B}^{p+q+1}_{p,q}(A) \xrightarrow{d} \cdots,$$

where pr denotes projection from the bicomplex A to the direct summand $\mathcal{B}^k_{p,q}(A)$ and $d = \partial + \bar{\partial}$.

By construction, we have isomorphisms

$$H^{p,q}_{\mathcal{BC}}(A) \cong H^{p+q}(\mathcal{B}^*_{p,q}(A)) \quad \text{and} \quad H^{p,q}_{\mathcal{A}}(A) \cong H^{p+q-1}(\mathcal{B}^*_{p,q}(A)).$$

The Bigolin complex is depicted in Figure 1.1.

Remark 1.3.2. Let A be a bicomplex. Denote by $A[p, q]$ the bicomplex given by

$$(A[p, q])^{r,s} = A^{r-p,s-q}.$$

Note that

$$\mathcal{B}_{p,q}(A) = \mathcal{B}_{0,0}(A[-p, -q]).$$

From now on, $\mathcal{B}$ will denote $\mathcal{B}_{0,0}$.

Lemma 1.3.3. *Consider on* $\mathrm{BiCo}_{\mathbf{k}}$ *the pluripotential model structure and on* $\mathrm{Ch}_{\mathbf{k}}$ *the projective model structure. The functor* $\mathcal{B}\colon \mathrm{BiCo}_{\mathbf{k}} \to \mathrm{Ch}_{\mathbf{k}}$ *preserves weak equivalences and fibrations.*

Proof. Corollary 1.25 of [Ste25a] states that if $f\colon A \to B$ is a pluripotential weak equivalence, then $H(f)$ is an isomorphism for any additive functor $H\colon \mathrm{BiCo}_{\mathbf{k}} \to V$, where V is an additive category and $H(P) = 0$ for every acyclic bicomplex P; that is, $H_{\mathcal{A}}(P) = H_{\mathcal{BC}}(P) = 0$ (see Definition 1.23 in [Ste25a]).

$$\begin{tikzcd}
 & & & A^{p,q+2} \arrow[r] & A^{p+1,q+2} & \iddots & \\
 & & & A^{p,q+1} \arrow[u] \arrow[r] & A^{p+1,q+1} \arrow[u] \arrow[r] & A^{p+2,q+1} & \\
 & & & A^{p,q} \arrow[u] \arrow[r] & A^{p+1,q} \arrow[u] \arrow[r] & A^{p+2,q} \arrow[u] & \\
A^{p-3,q-1} \arrow[r] & A^{p-2,q-1} \arrow[r] & A^{p-1,q-1} \arrow[ur, "\partial\bar{\partial}"] & & p+q & p+q+1 & \cdots \\
A^{p-3,q-2} \arrow[u] \arrow[r] & A^{p-2,q-2} \arrow[u] \arrow[r] & A^{p-1,q-2} \arrow[u] & p+q-1 & & & \\
\iddots & A^{p-2,q-3} \arrow[u] \arrow[r] & A^{p-1,q-3} \arrow[u] & p+q-2 & & & \\
 & & & \vdots & & &
\end{tikzcd}$$

Figure 1.1: The Bigolin complex

By [Ste25a], the functor $H^* \circ \mathcal{B}$, where H^* denotes taking cohomology of a cochain complex, satisfies this condition. It follows that $\mathcal{B}$ preserves weak equivalences. Moreover, in both model categories $\mathrm{BiCo}_{\mathbf{k}}$ and $\mathrm{Ch}_{\mathbf{k}}$, fibrations are simply (bi)degree-wise surjections. Since $\mathcal{B}$, viewed as a graded vector space, is given by the totalization of the restricted bicomplex, it preserves surjections. $\square$

It thus follows that if $\mathcal{B}$ is a right adjoint, then there is a Quillen adjunction between $\mathrm{BiCo}_{\mathbf{k}}$ and $\mathrm{Ch}_{\mathbf{k}}$. This is the content of Section 1.5.

1.4 Stairway to heaven

We introduce a family of minimal bicomplexes, which will serve as a minimal generalization of the shift functor, and state some properties. This family is the key ingredient for constructing the inflation functor in Section 1.5. Most of the content of this section can be found in [MSG].

Let us consider the following family of bicomplexes, $\{\mathsf{L}_n\}_{n\in\mathbb{Z}}$, where

$$\mathsf{L}_n^{p,q} = \begin{cases} \mathbf{k} \text{ if } p,q,n \geq 0 \text{ and } p+q=n \\ \mathbf{k} \text{ if } p,q > 0, n \geq 0 \text{ and } p+q=n+1, \\ \mathbf{k} \text{ if } p,q \leq 0, n < 0 \text{ and } p+q=n \\ \mathbf{k} \text{ if } p,q,n < 0 \text{ and } p+q=n-1 \\ 0 \text{ otherwise.} \end{cases} \tag{1.1}$$

Since $\mathsf{L}_n^{p,q}$ is either $\mathbf{k}$ or trivial, in the first case, $(p,q)_n$ will denote the unit in $\mathbf{k}$, while in the second case it will denote the zero element. The differentials in the bicomplex L_n are defined on the generators by

$$\partial(p,q)_n = q\cdot(p+1,q)_n \quad \text{and} \quad \bar\partial(p,q)_n = -p\cdot(p,q+1)_n.$$

Examples 1.4.1. We show some examples for different values of n.

- For $|n|=1$ we have:

$$\llcorner_1 = \begin{tikzcd} \mathbf{k} \arrow[r, "\cdot 1"] & \mathbf{k} \\ & \mathbf{k} \arrow[u, "\cdot(-1)"] \end{tikzcd} \qquad \llcorner_{-1} = \begin{tikzcd} \mathbf{k} & \\ \mathbf{k} \arrow[u, "\cdot 1"] \arrow[r, "\cdot(-1)"] & \mathbf{k} \end{tikzcd}.$$

- For $|n|=2$ we have:

$$\llcorner_2 = \begin{tikzcd} \mathbf{k} \arrow[r, "\cdot 2"] & \mathbf{k} & \\ & \mathbf{k} \arrow[u, "\cdot(-1)"] \arrow[r, "\cdot 1"] & \mathbf{k} \\ & & \mathbf{k} \arrow[u, "\cdot(-2)"] \end{tikzcd} \qquad \llcorner_{-2} = \begin{tikzcd} \mathbf{k} & & \\ \mathbf{k} \arrow[u, "\cdot 2"] \arrow[r, "\cdot(-1)"] & \mathbf{k} & \\ & \mathbf{k} \arrow[u, "\cdot 1"] \arrow[r, "\cdot(-2)"] & \mathbf{k} \end{tikzcd}.$$

- For $n=-3$ we have:

$$\llcorner_{-3} = \begin{tikzcd} \mathbf{k} & & & \\ \mathbf{k} \arrow[u, "\cdot 3"] \arrow[r, "\cdot(-1)"'] & \mathbf{k} & & \\ & \mathbf{k} \arrow[u, "\cdot 2"] \arrow[r, "\cdot(-2)"'] & \mathbf{k} & \\ & & \mathbf{k} \arrow[u, "\cdot 1"] \arrow[r, "\cdot(-3)"'] & \mathbf{k} \end{tikzcd}.$$

Remark 1.4.2. Note that the bicomplex $\llcorner_{-1}$ is precisely the bicomplex $\llcorner$ in the definition of the shift functor $\llcorner : \mathrm{BiCo}_{\mathbf{k}} \to \mathrm{BiCo}_{\mathbf{k}}$ defined in Section 1.1. Moreover, $\llcorner_{-n}$ is pluripotential weakly equivalent to the n-fold tensor product of $\llcorner$, that is,

$$\llcorner_{-n} \simeq \llcorner^{\otimes n}.$$

In the same way, $\llcorner_n$ is pluripotential weakly equivalent to the n-fold tensor product of $\urcorner$, that is,

$$\llcorner_n \simeq \urcorner^{\otimes n}.$$

The key advantage of $\llcorner_{-n}$ and $\llcorner_n$ is that they are minimal bicomplexes (i.e., $\partial\bar\partial = 0$), whereas $\llcorner^{\otimes n}$ and $\urcorner^{\otimes n}$ are not.

In the following, we will define maps involving the bicomplexes $\{\llcorner_n\}_{n\in\mathbb{Z}_{\leq 0}}$, which will be used to prove monoidality of the inflation functor and the Bigolin complex in Section 1.6 and in Koszul duality developed in Section 2.6.

Let us consider linear maps

$$\mathfrak{s}_n \colon \mathsf{L}_n \to \mathsf{L}_{n+1} \quad \text{and} \quad \bar{\mathfrak{s}}_n \colon \mathsf{L}_n \to \mathsf{L}_{n+1}$$

of bidegree $(1,0)$ and $(0,1)$, respectively, for $n \leq -1$ given by

$$\mathfrak{s}(p,q)_n = (-1)^{p+q+n}(p+1,q)_{n+1}, \quad \bar{\mathfrak{s}}(p,q)_n = (-1)^{p+q+n}(p,q+1)_{n+1}.$$

Remark 1.4.3. These maps satisfy the following

$$\partial\mathfrak{s}_n + \mathfrak{s}_n\partial = 0, \quad \bar{\partial}\bar{\mathfrak{s}}_n + \bar{\mathfrak{s}}_n\bar{\partial} = 0 \quad \text{and} \quad \partial\bar{\mathfrak{s}}_n + \bar{\partial}\mathfrak{s}_n + \bar{\mathfrak{s}}_n\partial + \mathfrak{s}_n\bar{\partial} = 0,$$

for all $n \leq -1$.

We introduce the following bicomplex maps

$$\iota_{k,\ell} \colon \mathsf{L}_{k+\ell} \to \mathsf{L}_k \otimes \mathsf{L}_\ell$$

of bidegree $(0,0)$ for every $k, \ell \in \mathbb{Z}_{\leq 0}$ given by

$$\iota_{k,\ell}(p,q)_{k+\ell} = \sum_{\substack{\alpha+\delta=p\\ \beta+\gamma=q}} (-1)^{(\alpha+\beta)(\delta+\gamma+\ell)}(\alpha,\beta)_k \otimes (\delta,\gamma)_\ell.$$

Proposition 1.4.4. *The maps are coassociative, in the sense that*

$$(\iota_{k_1,k_2} \otimes \mathrm{id})\iota_{k_1+k_2,k_3} = (\mathrm{id} \otimes \iota_{k_2,k_2})\iota_{k_1,k_2+k_3}$$

for any $k_1, k_2, k_3 \in \mathbb{Z}_{\leq 0}$.

Proof. On the one hand,

$$\begin{aligned}
(\iota_{r,s} \otimes \mathrm{id})\iota_{r+s,t}(i,j)_{r+s+t} &= \sum_{\substack{p_1+t_1=i\\ p_2+t_2=j}} (-1)^{d_p(d_t+t)}(\iota_{r,s} \otimes \mathrm{id})(p_1,p_2)_{r+s} \otimes (t_1,t_2)_t = \\
&= \sum_{\substack{p_1+t_1=i\\ p_2+t_2=j}} \sum_{\substack{r_1+s_1=p_1\\ r_2+s_2=p_2}} (-1)^{d_p(d_t+t)+d_r(d_s+s)}(r_1,r_2)_r \otimes (s_1,s_2)_s \otimes (t_1,t_2)_t = \\
&= \sum_{\substack{r_1+s_1+t_1=i\\ r_2+s_2+t_2=j}} (-1)^{(d_r+d_s)(d_t+t)+d_r(d_s+s)}(r_1,r_2)_r \otimes (s_1,s_2)_s \otimes (t_1,t_2)_t.
\end{aligned}$$

Where $d_x = x_1 + x_2$. On the other hand,

$$\begin{aligned}
(\mathrm{id} \otimes \iota_{s,t})\iota_{r,s+t}(i,j)_{r+s+t} &= \sum_{\substack{r_1+q_1=i\\ r_2+q_2=j}} (-1)^{d_r(d_q+q)}(\mathrm{id} \otimes \iota_{s,t})(r_1,r_2)_r \otimes (q_1,q_2)_q = \\
&= \sum_{\substack{r_1+q_1=i\\ r_2+q_2=j}} \sum_{\substack{s_1+t_1=q_1\\ s_2+t_2=q_2}} (-1)^{d_r(d_q+q)+d_s(d_t+t)}(r_1,r_2)_r \otimes (s_1,s_2)_s \otimes (t_1,t_2)_t = \\
&= \sum_{\substack{r_1+s_1+t_1=i\\ r_2+s_2+t_2=j}} (-1)^{d_r(d_s+s+d_t+t)+d_s(d_t+t)}(r_1,r_2)_r \otimes (s_1,s_2)_s \otimes (t_1,t_2)_t.
\end{aligned}$$

□

Proposition 1.4.5. *The maps*

$$\iota_{k,\ell}\colon \mathsf{L}_{k+\ell} \to \mathsf{L}_k \otimes \mathsf{L}_\ell$$

are pluripotential weak equivalences for all $k, \ell \leq 0$.

Proof. One can choose a splitting $\mathsf{L}_k \otimes \mathsf{L}_\ell \cong \mathsf{L}_{k+\ell} \oplus (\mathsf{L}_k \otimes \mathsf{L}_\ell)_{sq}$, where $(\mathsf{L}_k \otimes \mathsf{L}_\ell)_{sq}$ is given by the squares generated by the elements $(p_1, q_1)_k \otimes (p_2, q_2)_\ell$ where $p_1 + q_1 = k - 1$ and $p_2 + q_2 = \ell - 1$. Then, the map $\iota_{k,\ell}$ is just the inclusion of $\mathsf{L}_{k+\ell}$ into $\mathsf{L}_k \otimes \mathsf{L}_\ell$. Since the squares are homotopically trivial, we obtain that $\iota_{k,\ell}$ is a pluripotential weak equivalence. □

Proposition 1.4.6. *The following hold*

1. $(\mathfrak{s}_{k-1} \otimes \mathrm{id})\iota_{k-1,\ell} = (-1)^k(\mathrm{id} \otimes \mathfrak{s}_{\ell-1})\iota_{k,\ell-1} = \iota_{k,\ell}\mathfrak{s}_{k+\ell-1}$,
2. $(\bar{\mathfrak{s}}_{k-1} \otimes \mathrm{id})\iota_{k-1,\ell} = (-1)^k(\mathrm{id} \otimes \bar{\mathfrak{s}}_{\ell-1})\iota_{k,\ell-1} = \iota_{k,\ell}\bar{\mathfrak{s}}_{k+\ell-1}$.

Proof. Let us prove (1), (2) is analogous. On the one hand, we have

$$\begin{aligned}(\mathfrak{s} \otimes \mathrm{id})\iota_{k-1,\ell}(p,q)_{k+\ell-1} &= \sum_{\substack{p_1+p_2=p\\ q_1+q_2=q}} (-1)^{d_1+k-1+d_1(d_2+\ell)}(p_1+1, q_1)_k \otimes (p_2, q_2)_\ell \\ &= \sum_{\substack{p_1+p_2=p+1\\ q_1+q_2=q}} (-1)^{d_1+k+d_2+\ell+d_1(d_2+\ell)}(p_1, q_1)_k \otimes (p_2, q_2)_\ell,\end{aligned}$$

where $d_i = p_i + q_i$. On the other hand,

$$\begin{aligned}(\mathrm{id} \otimes \mathfrak{s})\iota_{k,\ell-1}(p,q)_{k+\ell-1} &= \sum_{\substack{p_1+p_2=p\\ q_1+q_2=q}} (-1)^{d_2+\ell-1+d_1(d_2+\ell-1)+d_1}(p_1, q_1)_k \otimes (p_2+1, q_2)_\ell \\ &= \sum_{\substack{p_1+p_2=p+1\\ q_1+q_2=q}} (-1)^{d_1+d_2+\ell+d_1(d_2+\ell)}(p_1, q_1)_k \otimes (p_2, q_2)_\ell.\end{aligned}$$

Finally,

$$\iota_{k,\ell}\mathfrak{s}(p,q)_{k+\ell-1} = \sum_{\substack{p_1+p_2=p+1\\ q_1+q_2=q}} (-1)^{d_1+d_2+k+\ell+d_1(d_2+\ell)}(p_1, q_1)_k \otimes (p_2, q_2)_\ell.$$ □

Define a generalized map $\iota_{k_0,\dots,k_r}\colon \mathsf{L}_{k_0+\cdots+k_r} \to \mathsf{L}_{k_0} \otimes \cdots \otimes \mathsf{L}_{k_r}$ for $k_0, \dots, k_r \in \mathbb{Z}_{\leq 0}$ by recursively applying $\iota_{k,\ell}$, which gives

$$\iota_{k_0,\dots,k_r}(p,q)_n = \sum_{\substack{p_0+\cdots+p_r=p\\ q_0+\cdots+q_r=q}} (-1)^\varepsilon (p_0, q_0)_{k_0} \otimes \cdots \otimes (p_r, q_r)_{k_r},$$

where $\varepsilon := \sum_{j=1}^r \left(\sum_{i=0}^{j-1}(p_i+q_i)\right)(p_j + q_j + k_j)$ and $n = k_0 + \cdots + k_r$.

The following is a consequence of Proposition 1.4.5.

Proposition 1.4.7. *The maps $\iota_{k_0,\dots,k_r}$ are pluripotential weak equivalences for every $k_0, \dots, k_r \in \mathbb{Z}_{\leq 0}$.*

1.5 The inflation functor

In this section we introduce the *inflation* functor from cochain complexes to bicomplexes and show it is the left adjoint of the Bigolin complex. Most of the content of this section can be found in [MSG].

We define a functor

$$\mathrm{Inf} : \mathrm{Ch}_{\mathbf{k}} \to \mathrm{BiCo}_{\mathbf{k}}.$$

Given a chain complex (C, d) we set

$$\mathrm{Inf}(C) = \left(\bigoplus_{n \in \mathbb{Z}} \mathsf{L}_n \otimes C^n, \partial, \bar{\partial} \right),$$

where we consider C^n as a bicomplex sitting in bidegree $(0, 0)$. The differentials ∂ and $\bar{\partial}$ are defined by:

$$\begin{aligned} \partial((p,q)_n \otimes c) &= q(p+1,q)_n \otimes c + (-1)^{p+q+n}(p+1,q)_{n+1} \otimes dc, \\ \bar{\partial}((p,q)_n \otimes c) &= -p(p,q+1)_n \otimes c + (-1)^{p+q+n}(p,q+1)_{n+1} \otimes dc. \end{aligned} \tag{1.2}$$

Remark 1.5.1. The differentials of $\mathrm{Inf}(C)$ correspond to

$$\partial = \partial_{\mathsf{L}} \otimes \mathrm{id} + \mathfrak{s} \otimes d \quad \text{ and } \quad \bar{\partial} = \bar{\partial}_{\mathsf{L}} \otimes \mathrm{id} + \bar{\mathfrak{s}} \otimes d,$$

see Section 1.4 for the definition of $\mathfrak{s}$ and $\bar{\mathfrak{s}}$. By Remark 1.4.3, $\mathrm{Inf}(C)$ forms a bicomplex.

We refer to this functor as the *inflation functor*, which is depicted in Figure 1.2.

$$\begin{tikzcd}
 & & & C^2 \arrow[r, shift left, blue] \arrow[r, shift right, blue] & C^2 \oplus C^3 & \iddots \\
 & & & C^1 \arrow[u] \arrow[r, shift left, blue] \arrow[r, shift right, blue] & C^1 \oplus C^2 \arrow[u, shift left, blue] \arrow[u, shift right] \arrow[r, shift left, blue] \arrow[r, shift right, blue] & C^2 \oplus C^3 \\
C^{-3} \arrow[r] & C^{-2} \arrow[r] & C^{-1} \arrow[r] & C^0 \arrow[u] \arrow[r] & C^1 \arrow[u, shift left, blue] \arrow[u, shift right] \arrow[r] & C^2 \arrow[u, shift left, blue] \arrow[u, shift right] \\
C^{-3} \oplus C^{-4} \arrow[u, shift left, blue] \arrow[u, shift right] \arrow[r, shift left, blue] \arrow[r, shift right, blue] & C^{-2} \oplus C^{-3} \arrow[u, shift left, blue] \arrow[u, shift right] \arrow[r, shift left, blue] \arrow[r, shift right, blue] & C^{-1} \oplus C^{-2} \arrow[u, shift left, blue] \arrow[u, shift right] \arrow[r, shift left, blue] \arrow[r, shift right, blue] & C^{-1} \arrow[u] & & \\
\iddots & C^{-3} \oplus C^{-4} \arrow[u, shift left, blue] \arrow[u, shift right] \arrow[r, shift left, blue] \arrow[r, shift right, blue] & C^{-2} \oplus C^{-3} \arrow[u, shift left, blue] \arrow[u, shift right] \arrow[r, shift left, blue] \arrow[r, shift right, blue] & C^{-2} \arrow[u] & & \\
 & \iddots & C^{-3} \oplus C^{-4} \arrow[u, shift left, blue] \arrow[u, shift right] \arrow[r, shift left, blue] \arrow[r, shift right, blue] & C^{-3} \arrow[u] & &
\end{tikzcd}$$

Figure 1.2: The inflation functor

The blue arrows denote the differentials of the bicomplexes $\mathsf{L}_n \otimes C^n$ and the black arrows are the "connecting" differentials of the inflation functor given by the differential d of C.

Examples 1.5.2. We compute the inflation functor on simplicial cochain complexes:

- Consider the cochain complex $C_*(\Delta^1)$

$$\underset{-1}{\mathbf{k}\langle c\rangle} \xrightarrow{d} \underset{0}{\mathbf{k}\langle c_1\rangle \oplus \mathbf{k}\langle c_2\rangle} \qquad d(c) = c_2 - c_1.$$

  $\mathrm{Inf}(C_*(\Delta^1))$ is given by

$$\begin{tikzcd} \mathbf{k} \arrow[r, "d"] & \mathbf{k}\oplus\mathbf{k} \\ \mathbf{k} \arrow[u, "id"] \arrow[r, "-id"] & \mathbf{k} \arrow[u, "d"'] \end{tikzcd},$$

  so that $\partial\bar{\partial}c = c_2 - c_1$.

- Consider the cochain complex $C := C_*(\Delta^2)$,

$$\underset{-2}{\mathbf{k}\langle[012]\rangle} \xrightarrow{d_2} \underset{-1}{\mathbf{k}\langle[01]\rangle \oplus \mathbf{k}\langle[02]\rangle \oplus \mathbf{k}\langle[12]\rangle} \xrightarrow{d_1} \underset{0}{\mathbf{k}\langle[0]\rangle \oplus \mathbf{k}\langle[1]\rangle \oplus \mathbf{k}\langle[2]\rangle}.$$

  $\mathrm{Inf}(C_*(\Delta^2))$ is given by

$$\begin{tikzcd} \mathbf{k} \arrow[r, "d_2"] & \mathbf{k}^{\oplus 3} \arrow[r, "d_1"] & \mathbf{k}^{\oplus 3} \\ \mathbf{k} \arrow[u, "\cdot 2"] \arrow[r, shift left, "-d_2"] \arrow[r, shift right, "\cdot(-1)"'] & \mathbf{k}\oplus\mathbf{k}^{\oplus 3} \arrow[u, shift left, "\cdot 1"] \arrow[u, shift right, "d_2"'] \arrow[r, shift left, "d_2"] \arrow[r, shift right, "\cdot(-1)"'] & \mathbf{k}^{\oplus 3} \arrow[u, "d_1"] \\ & \mathbf{k} \arrow[u, shift left, "\cdot 1"] \arrow[u, shift right, "-d_2"'] \arrow[r, "\cdot(-2)"'] & \mathbf{k} \arrow[u, "d_2"] \end{tikzcd}.$$

  Here, the blue bicomplex is $\mathsf{L}_{-2} \otimes C_{-2}$, the red bicomplex is $\mathsf{L}_{-1} \otimes C_{-1}$, and the orange bicomplex is $\mathsf{L}_0 \otimes C_0 = \mathbf{k} \otimes C_0 \cong C_0$, which sits in bidegree $(0,0)$.

In both examples, $H_{\mathcal{A}} = 0$ and $H_{\mathcal{BC}} = 0$, so they are acyclic bicomplexes in the pluripotential sense.

Observe that Inf preserves direct sums and that given an injection of chain complexes $D \hookrightarrow C$, the canonical morphism

$$\begin{aligned} \mathrm{Inf}(C)/\mathrm{Inf}(D) &\to \mathrm{Inf}(C/D) \\ [(i,j)_n \otimes c] &\mapsto (i,j)_n \otimes [c] \end{aligned}$$

is an isomorphism. So Inf commutes with colimits. As $\mathrm{Ch}_{\mathbf{k}}$ and $\mathrm{BiCo}_{\mathbf{k}}$ are locally presentable, it follows that Inf admits a right adjoint. In the following theorem, we show that the inflation functor and the Bigolin complex form an adjunction. We present two proofs: the first is more elegant, while the second introduces constructions that will be used later.

Theorem 1.5.3. *The inflation functor* Inf *is left adjoint to the Bigolin complex* $\mathcal{B}$*.*

Proof 1. Denote by $\langle\delta\rangle$ the $\mathbf{k}$-linear category with objects $n \in \mathbb{Z}$ and morphisms

$$\langle\delta\rangle(m,n) = \begin{cases} \mathbf{k} & \text{if } m-n = 0 \text{ or } 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, cochain complexes are linear functors on $\langle\delta\rangle$ with values in $\mathbf{k}$-Mod, Ch $= [\langle\delta\rangle, \mathbf{k}\text{-Mod}]$. Similarly, denote by $\langle\partial, \overline{\partial}\rangle$ the category whose objects are pairs $(m, n) \in \mathbb{Z}^2$ and morphisms are

$$\langle\partial, \overline{\partial}\rangle((m,n),(m',n')) = \begin{cases} \partial\mathbf{k} & \text{if } m - m' = 1 \text{ and } n = n', \\ \overline{\partial}\mathbf{k} & \text{if } n - n' = 1 \text{ and } m = m', \\ \mathrm{id}\mathbf{k} & \text{if } n = n' \text{ and } m = m', \\ \partial\bar{\partial}\mathbf{k} & \text{if } m - m' = 1 \text{ and } n - n' = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define also the composition law so as to have $\partial\overline{\partial} = -\overline{\partial}\partial$. Then, bicomplexes are presheaves on $\langle\partial, \overline{\partial}\rangle$ with values in $\mathbf{k}$-Mod.

Denote by $\iota\colon \langle\delta\rangle \to \mathrm{Ch}_{\mathbf{k}}$ the Yoneda embedding and define $F = \mathrm{Inf} \circ \iota$. Since both Ch and $\mathrm{BiCo}_{\mathbf{k}}$ are presheaf categories and Inf preserves colimits, the right adjoint R to Inf is given by

$$R(B)_n = \mathrm{Hom}_{\mathrm{BiCo}_{\mathbf{k}}}(F(n), B), \qquad \text{for } B \text{ in } \mathrm{BiCo}_{\mathbf{k}}.$$

Note that $F(n) = \mathrm{Inf}(D^n)$, where the disk D^n is the cochain complex

$$\cdots \to 0 \to \underset{n}{\mathbf{k}} \to \underset{n+1}{\mathbf{k}} \to 0 \to \cdots .$$

As a bigraded vector space, $\mathrm{Inf}(D^n) = (\llcorner_n \otimes \mathbf{k}) \oplus (\llcorner_{n+1} \otimes \mathbf{k})$ and is freely generated by the elements in the lowest degree. This implies that a morphism in $\mathrm{Hom}_{\mathrm{BiCo}_{\mathbf{k}}}(F(n), B)$ is determined by the image of the copies of $\mathbf{k}$ in $\mathrm{Inf}(D^n)$ of lowest bidegrees. That is, it is determined by a choice of elements in

$$\bigoplus_{i\geq 0}^{n} B^{i,n-i} \qquad \text{for } n \geq 0$$
$$\bigoplus_{i=n}^{-1} B^{i,n-i-1} \qquad \text{for } n < 0.$$

It follows that, as graded $\mathbf{k}$-modules, $R(B) = \mathcal{B}(B)$. Denoting by δ the generating morphism of $\langle\delta\rangle(n+1, n)$, one has that $F(\delta)$ simply projects onto $(\llcorner_{n+1} \otimes \mathbf{k})$. The differential of $R(B)$ is given by

$$\begin{aligned} \mathrm{Hom}_{\mathrm{BiCo}_{\mathbf{k}}}(F(n), B) &\xrightarrow{d} \mathrm{Hom}_{\mathrm{BiCo}_{\mathbf{k}}}(F(n+1), B) \\ f &\mapsto f \circ F(\delta). \end{aligned}$$

For $n = -1$, we have

$$F(0) = \begin{tikzcd} \mathbf{k} \arrow[r, "id"] & \mathbf{k} \\ \mathbf{k} \arrow[u, "id"] \arrow[r, "id"] & \mathbf{k} \arrow[u, "-id"'] \end{tikzcd} \qquad F(-1) = \begin{tikzcd} \mathbf{k} \arrow[r, "id"] & \mathbf{k} \\ \mathbf{k} \arrow[u, "id"] \arrow[r, "-id"] & \mathbf{k} \arrow[u, "id"'] \end{tikzcd}$$

where the bottom left corners of $F(0)$ and $F(-1)$ sit in bidegree $(0,0)$ and $(-1,-1)$, respectively.

A morphism $g \in \mathrm{Hom}_{\mathrm{BiCo}_{\mathbf{k}}}(F(0), B)$ is determined by $g^{0,0}$ since it commutes with differentials. Likewise, $f \in \mathrm{Hom}_{\mathrm{BiCo}_{\mathbf{k}}}(F(-1), B)$ is determined by $f^{\text{-}1,\text{-}1}$, so we can make the identifications

$$R^0(B) = \mathrm{Hom}_{\mathrm{BiCo}_{\mathbf{k}}}(F(0), B) \cong B^{0,0}, \qquad R^{-1}(B) = \mathrm{Hom}_{\mathrm{BiCo}_{\mathbf{k}}}(F(-1), B) \cong B^{-1,-1}.$$

Note that $F(\delta) : F(0) \to F(-1)$ projects onto the copy of $\mathbf{k}$ in bidegree $(0,0)$. The image of $f \in \mathrm{Hom}_{\mathrm{BiCo}_{\mathbf{k}}}(F(-1), B)$ by the differential $d : R^{-1}(B) \to R^0(B)$ is then given by $(f \circ F(\delta))^{0,0} = f^{0,0}$. But $f^{0,0} = \partial\bar{\partial} f^{\text{-}1,\text{-}1}$. Hence, $d(f^{\text{-}1,\text{-}1}) = \partial\bar{\partial} f^{\text{-}1,\text{-}1}$, which is the differential of the complex $\mathcal{B}$.

For $n < -1$ the differential of $R(B)$ is given by

$$d(b^{n,\text{-}1}, \dots, b^{\text{-}1,n}) = \sum_{p+q=n-1} \frac{(p+1)\partial b^{p,q} + (q+1)\bar{\partial} b^{p,q}}{-n}$$

and for $n \geq 0$ is given by

$$d(b^{0,n}, \dots, b^{n,0}) = \sum_{p+q=n} \frac{(p+1)\partial b^{p,q} + (q+1)\bar{\partial} b^{p,q}}{n}.$$

We define a natural isomorphism from $f \colon R \to \mathcal{B}$ by

$$f^n(b^{n,\text{-}1}, \dots, b^{\text{-}1,n}) = (-1)^{n+1}(\mathrm{L}(\text{-}n,1)b^{n,\text{-}1}, \mathrm{L}(\text{-}n,2)b^{n+1,\text{-}2}, \dots, \mathrm{L}(1,\text{-}n)b^{\text{-}1,n}),$$

for $n < 0$, and by

$$f^n(b^{0,n}, \dots, b^{n,0}) = (\binom{n}{0} b^{0,n}, \binom{n}{1} b^{1,n}, \dots, \binom{n}{n} b^{n,0}),$$

for $n \geq 0$. Here, $\binom{k}{n}$ denote the binomial coefficients whereas $\mathrm{L}(k,\ell)$ denote the entries of the Leibniz harmonic triangle [OEI26], explicitly given by

$$\mathrm{L}(k,\ell) = \frac{1}{\ell\binom{k}{\ell}}. \qquad \square$$

Proof 2. We prove the adjunction by explicitly constructing the unit and the counit. We begin by defining the unit of the adjunction

$$\eta : \mathrm{Id}_{\mathrm{Ch}_{\mathbf{k}}} \to \mathcal{B}\mathrm{Inf}.$$

Note that, for a cochain complex C,

$$\mathcal{B}\mathrm{Inf}(C)^n = \begin{cases} \displaystyle\bigoplus_{\substack{p+q=n \\ p,q\geq 1}} (p,q)_{n-1} \otimes C^{n-1} \;\oplus\; \bigoplus_{\substack{p+q=n \\ p,q\geq 0}} (p,q)_n \otimes C^n & \text{if } n \geq 0, \\ \displaystyle\bigoplus_{\substack{p+q=n-1 \\ p,q\leq -1}} (p,q)_n \otimes C^n \;\oplus\; \bigoplus_{\substack{p+q=n-1 \\ p,q\leq -1}} (p,q)_{n-1} \otimes C^{n-1} & \text{if } n < 0. \end{cases}$$

We present in Figure 1.3 a picture of $\mathcal{B}\mathrm{Inf}(C)$. The black piece together with the gray one constitute $\mathrm{Inf}(C)$, whereas the "totalization" of the black part constitutes $\mathcal{B}\mathrm{Inf}(C)$. The numbers on the right hand side denote the degree of the Bigolin complex.

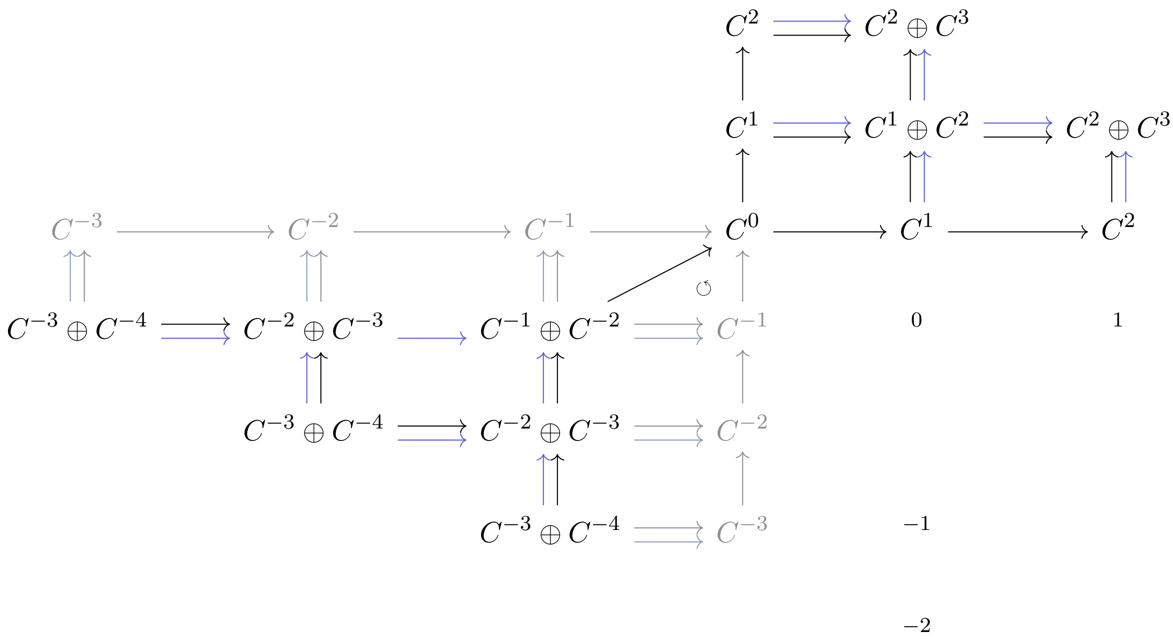


Figure 1.3: The complex $\mathcal{B}\mathrm{Inf}(C)$

We define:

$$\eta_C(c) = \begin{cases} \displaystyle\sum_{\substack{p+q=n \\ p+q\geq 0}} \binom{n}{p} (p,q)_n \otimes c & \text{if } n \geq 0, \\ \displaystyle\sum_{\substack{p+q=n-1 \\ p,q\leq -1}} (-1)^n \mathrm{L}(-n,-p)(p,q)_n \otimes c & \text{if } n < 0. \end{cases}$$

Here, $\binom{k}{n}$ denote the binomial coefficients and $\mathrm{L}(k,\ell)$ denote the entries of the Leibniz harmonic triangle.

We now define the counit of the adjunction

$$\varepsilon : \mathrm{Inf}\mathcal{B} \to \mathrm{Id}_{\mathrm{BiCo}_{\mathbf{k}}}.$$

Note that, for a bicomplex A,

$$\mathrm{Inf}\mathcal{B}(A)^{p,q} = \begin{cases} \displaystyle\bigoplus_{\substack{r+s=p+q \\ r,s\geq 0}} (p,q)_{p+q} \otimes A^{r,s} \oplus \bigoplus_{\substack{r+s=p+q-1 \\ r,s\geq 0}} (p,q)_{p+q-1} \otimes A^{r,s} & \text{if } p,q \geq 0, \\ \displaystyle\bigoplus_{\substack{r+s=p+q-1 \\ r,s\leq -1}} (p,q)_{p+q} \otimes A^{r,s} \oplus \bigoplus_{\substack{r+s=p+q \\ r,s\leq -1}} (p,q)_{p+q+1} \otimes A^{r,s} & \text{if } p,q < 0. \end{cases}$$

We present in Figure 1.4 a picture of $\mathrm{Inf}\mathcal{B}(A)$. Again, the blue arrows denote the differentials of the bicomplexes $\mathsf{L}_n \otimes \mathcal{B}(A)^n$. The black arrows are given by the differential of the Bigolin complex.

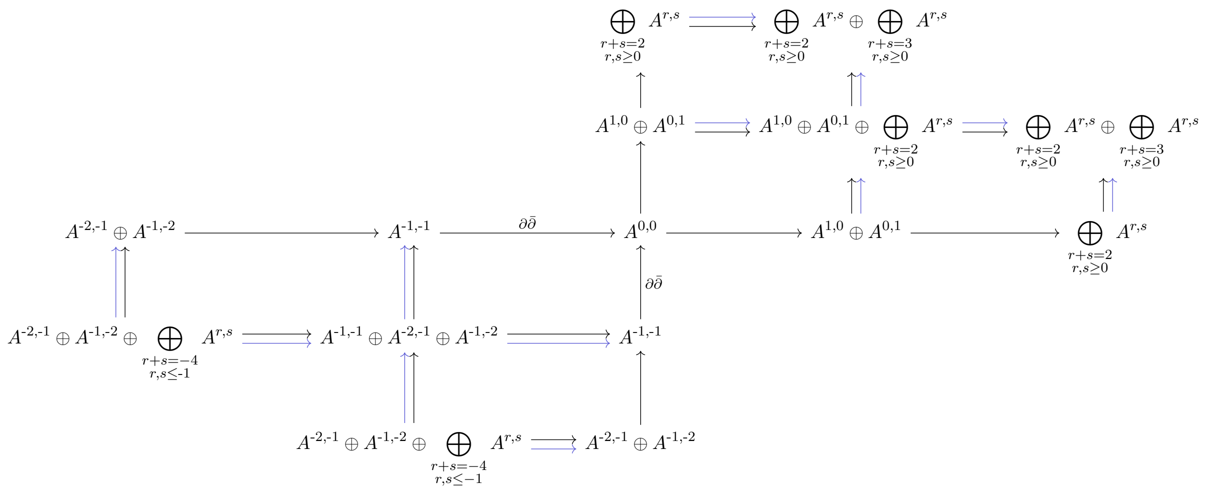


Figure 1.4: The bicomplex $\mathrm{Inf}\mathcal{B}(A)$

We define:

$$\varepsilon_A^{p,q}((p,q)_{p+q} \otimes (a^{p+q,0}, \ldots, a^{0,p+q}) + (p,q)_{p+q-1} \otimes (a^{p+q-1,0}, \ldots, a^{0,p+q-1})) = \\ = p\mathrm{L}(p+q,p)a^{pq} + \mathrm{L}(p+q,p)\partial a^{p-1,q} - \mathrm{L}(p+q,q)\bar{\partial} a^{p,q-1},$$

if $p, q \geq 0$ and

$$\varepsilon_A^{p,q}((p,q)_{p+q+1} \otimes (a^{p+q+1,-1}, \ldots, a^{-1,p+q+1}) + (p,q)_{p+q} \otimes (a^{p+q,-1}, \ldots, a^{-1,p+q})) = \\ = (-1)^{p+q+1}(\frac{1}{\mathrm{L}(-p-q-1,-p)}a^{pq} + \binom{-p-q-1}{-q-1}\partial a^{p-1,q} - \binom{-p-q-1}{-p-1}\bar{\partial} a^{p,q-1}),$$

if $p, q < 0$.

It remains to verify that

$$\mathrm{Inf} \xrightarrow{\mathrm{Inf}\eta} \mathrm{Inf}\mathcal{B}\mathrm{Inf} \xrightarrow{\varepsilon\mathrm{Inf}} \mathrm{Inf}$$

and

$$\mathcal{B} \xrightarrow{\eta\mathcal{B}} \mathcal{B}\mathrm{Inf}\mathcal{B} \xrightarrow{\mathcal{B}\varepsilon} \mathcal{B}$$

are the identity morphisms 1_{Inf} and $1_{\mathcal{B}}$, respectively.

Let A be a bicomplex, and let $\bar{a} = (a^{n,0}, a^{n-1,1}, \ldots, a^{0,n})$ be an element in $\mathcal{B}(A)^n$ for $n \geq 0$. The map

$$\eta_{\mathcal{B}} \colon \mathcal{B}(A) \to \mathcal{B}\mathrm{Inf}\mathcal{B}(A)$$

gives

$$\bar{a} \mapsto \sum_{\substack{p+q=n \\ p,q\geq 0}} \binom{n}{p} (p,q)_n \otimes \bar{a}.$$

Applying

$$\mathcal{B}(\varepsilon) \colon \mathcal{B}\mathrm{Inf}\mathcal{B}(A) \to \mathcal{B}(A)$$

gives

$$\sum_{\substack{p+q=n\\p,q\geq 0}} \binom{n}{p}(p,q)_n \otimes (a^{0,n},\dots,a^{n,0}) \mapsto \sum_{\substack{p+q=n\\p,q\geq 0}} \binom{n}{q}\frac{1}{\binom{n}{p}}a^{p,q} = \overline{a}.$$

Now let $\overline{a} = (a^{n,-1}, a^{n+1,-2}, \dots, a^{-1,n})$ be an element in $\mathcal{B}(A)^n$ for $n < 0$. The map

$$\eta_{\mathcal{B}} \colon \mathcal{B}(A) \to \mathcal{B}\mathrm{Inf}\mathcal{B}(A)$$

gives

$$\overline{a} \mapsto \sum_{\substack{p+q=n-1\\p,q\leq -1}} (-1)^n \mathrm{L}(-n,-p)(p,q)_n \otimes \overline{a}.$$

Applying

$$\mathcal{B}(\varepsilon)\colon \mathcal{B}\mathrm{Inf}\mathcal{B}(A) \to \mathcal{B}(A)$$

gives

$$\sum_{\substack{p+q=n-1\\p,q\leq -1}} (-1)^n \mathrm{L}(-n,-p)(p,q)_n \otimes \overline{a} \mapsto \sum_{\substack{p+q=n-1\\p,q\leq -1}} (-1)^{n+p+q+1}\frac{\mathrm{L}(-n,-p)}{\mathrm{L}(-p-q-1,-p)}a^{p,q} = \overline{a}.$$

We leave it as an exercise to the reader to show that the unit is a map of chain complexes, that the counit is a map of bicomplexes, and that the composition $\mathrm{Inf} \xrightarrow{F\eta} \mathrm{Inf}\mathcal{B}\mathrm{Inf} \xrightarrow{\varepsilon\mathrm{Inf}} \mathrm{Inf}$ is the identity map. Our sympathies go to the brave souls who attempt the exercise. □

Remark 1.5.4. The binomial coefficients and the entries of the Leibniz harmonic triangle are related by the formula

$$\mathrm{L}(r,c) = \frac{1}{c\binom{r}{c}}.$$

Moreover, both Pascal and Leibniz triangles are defined by the same recursive rule: each entry in the triangle is obtained as the sum of the two nearest entries in the row below, see Figure 1.5.

$$\begin{array}{ccccccccc}
 & & \vdots & & & & \vdots & & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & 1 & & 3 & & 3 & & 1 & \\
 & & 1 & & 2 & & 1 & & \\
 & & & 1 & & 1 & & & \\
 & & & & 1 & & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \\
 & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\
 & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\
\frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} \\
 & & \vdots & & & & \vdots & &
\end{array}$$

Figure 1.5: Pascal and Leibniz triangles

Remark 1.5.5. Note that, for bicomplexes concentrated in the first quadrant, the Bigolin complex is just the usual totalization of bicomplexes. Therefore, in this particular setting, the inflation functor is left adjoint to totalization.

Consider $\mathrm{Ch}_{\mathbf{k}}$ endowed with the projective model structure and $\mathrm{BiCo}_{\mathbf{k}}$ with the pluripotential model structure.

Theorem 1.5.6. *There is a Quillen adjunction*

$$\mathrm{Inf} : \mathrm{Ch}_{\mathbf{k}} \rightleftarrows \mathrm{BiCo}_{\mathbf{k}} : \mathcal{B}\ .$$

Proof. By Lemma 1.3.3, the functor $\mathcal{B}$, which is the right adjoint, preserves weak equivalences and fibrations, it follows that $\mathrm{Inf} \dashv \mathcal{B}$ is a Quillen adjunction. ☐

Proposition 1.5.7. *The inflation functor preserves weak equivalences. Moreover, it is conservative.*

Proof. The first part of the proposition is a corollary of Theorem 1.5.6. The second part can be proven as follows: let $f\colon C \to D$ be a morphism of cochain complexes such that $\mathrm{Inf}(f)$ is a pluripotential weak equivalence. By Theorem 1.21 of [Ste25a], it is a quasi-isomorphism with respect to the differential ∂ and $\bar{\partial}$. Note that the cochain complex $\mathrm{Inf}(C)^{*,0}$ is isomorphic to the cochain complex C^*, this is true for any cochain complex C. It follows that $f = \mathrm{Inf}(f)_{|\mathrm{Inf}(C)^{*,0}} : \mathrm{Inf}(C)^{*,0} \to \mathrm{Inf}(D)^{*,0}$ is a quasi-isomorphism of cochain complexes. ☐

1.6 Monoidal properties

In this section, we show that the inflation functor is colax symmetric and that the Bigolin complex is lax symmetric. The monoidality of the inflation functor will be used in Koszul duality.

Denote by $\mathrm{BiCo}_{\mathbf{k}}^{\urcorner}$ the subcategory of bicomplexes concentrated in the third quadrant, that is, $B^{p,q} = 0$ for $p > 0$ or $q > 0$. Note that $\mathrm{Inf}(C)$ for a cochain complex C in $\mathrm{Ch}_{\mathbf{k}}^{\leq 0}$ lies in $\mathrm{BiCo}_{\mathbf{k}}^{\urcorner}$. It follows directly from Theorem 1.5.3 that there is an adjunction

$$\mathrm{Inf} : \mathrm{Ch}_{\mathbf{k}}^{\leq 0} \rightleftarrows \mathrm{BiCo}_{\mathbf{k}}^{\urcorner} : \mathcal{B}\ .$$

Theorem 1.6.1. *The functor* $\mathrm{Inf}\colon \mathrm{Ch}_{\mathbf{k}}^{\leq 0} \to \mathrm{BiCo}_{\mathbf{k}}^{\urcorner}$ *is symmetric colax monoidal with structure morphisms given by*

- $\phi = \mathrm{id} : \mathrm{Inf}(\mathbf{k}) \to \mathbf{k}$*, and*
- *for all C and D in* $\mathrm{Ch}_{\mathbf{k}}^{\leq 0}$,

$$\phi_{C,D} : \mathrm{Inf}(C \otimes D) \to \mathrm{Inf}(C) \otimes \mathrm{Inf}(D), \tag{1.3}$$

defined by

$$\phi_{C,D}((i,j)_{k+\ell} \otimes c \otimes d) = \sum_{\substack{\alpha+\delta=i \\ \beta+\gamma=j}} (-1)^{(\alpha+\beta)(\delta+\gamma+\ell)} (\alpha,\beta)_k \otimes c \otimes (\delta,\gamma)_\ell \otimes d,$$

for $c \in C^k$ and $d \in D^\ell$.

Moreover, for all C and D in $\mathrm{Ch}_{\mathbf{k}}^{\leq 0}$, the structure morphism $\phi_{C,D}$ is a pluripotential weak equivalence.

Proof. Let $C, D \in \mathrm{Ch}_{\mathbf{k}}^{\leq 0}$, first note that

$$\mathrm{Inf}(C \otimes D) = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{k+\ell=n} \llcorner_n \otimes C^k \otimes D^\ell$$

and

$$\mathrm{Inf}(C) \otimes \mathrm{Inf}(D) = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{k+\ell=n} \llcorner_k \otimes C^k \otimes \llcorner_\ell \otimes D^\ell$$

We now use the maps

$$\iota_{k,\ell} : \llcorner_{k+\ell} \to \llcorner_k \otimes \llcorner_\ell,$$

defined in Section 1.4 for every $k, \ell \in \mathbb{Z}_{\leq 0}$, to define the following maps:

$$\tau \circ (\iota_{k,\ell} \otimes id^{\otimes 2}) : \llcorner_{k+\ell} \otimes C^k \otimes D^\ell \xrightarrow{\iota_{k,\ell}} \llcorner_k \otimes \llcorner_\ell \otimes C^k \otimes D^\ell \underset{\cong}{\xrightarrow{\tau}} \llcorner_k \otimes C^k \otimes \llcorner_\ell \otimes D^\ell,$$

which yield a map of bicomplexes

$$\phi_{C,D} : \mathrm{Inf}(C \otimes D) \to \mathrm{Inf}(C) \otimes \mathrm{Inf}(D)$$

by Remark 1.5.1 and Proposition 1.4.6. Note that there are no signs involved in τ since we consider C^p a bicomplex sitting in degree $(0,0)$. The monoidal unit in $\mathrm{BiCo}_{\mathbf{k}}$ and in $\mathrm{Ch}_{\mathbf{k}}$ is $\mathbf{k}$. So we can define a morphism

$$\phi = \mathrm{id} : \mathrm{Inf}(\mathbf{k}) \to \mathbf{k},$$

and the unitality condition follows immediately.

It remains to check coassociativity, that is:

$$(\phi_{C,D} \otimes \mathrm{id}) \circ \phi_{C\otimes D,E} = (\mathrm{id} \otimes \phi_{D,E}) \circ \phi_{C,D\otimes E} : \mathrm{Inf}(C \otimes D \otimes E) \to \mathrm{Inf}(C) \otimes \mathrm{Inf}(D) \otimes \mathrm{Inf}(E).$$

We need to show

$$(\iota_{r,s} \otimes \mathrm{id}) \circ \iota_{r+s,t} = (\mathrm{id} \otimes \iota_{s+t}) \circ \iota_{r,s+t}$$

for all $r, s, t \in \mathbb{Z}_{\leq 0}$, but this was proven in Proposition 1.4.4. To prove it is symmetric, we want to check that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Inf}(C \otimes D) \arrow[r, "\tau_1", "\cong"'] \arrow[d, "\phi_{C,D}"'] & \mathrm{Inf}(D \otimes C) \arrow[d, "\phi_{D,C}"] \\
\mathrm{Inf}(C) \otimes \mathrm{Inf}(D) \arrow[r, "\tau_2", "\cong"'] & \mathrm{Inf}(D) \otimes \mathrm{Inf}(C) .
\end{tikzcd}$$

Let $(i,j)_{k+\ell} \otimes c \otimes d \in \mathrm{Inf}(C \otimes D)$ where $\mid c \mid = k$ and $\mid d \mid = \ell$. On the one hand, we have:

$$\phi_{D,C} \circ \tau_1 = \sum_{\substack{p_1+q_1=i \\ p_2+q_2=j}} (-1)^{k\ell + d_q(d_p+k)} (q_1, q_2)_\ell \otimes d \otimes (p_1, p_2)_k \otimes c,$$

where $d_x = x_1 + x_2$. On the other:

$$\tau_2\phi_{C,D} = \sum_{\substack{p_1+p_2=p\\ q_1+q-2=q}} (-1)^{d_p(d_q+\ell)+d_pd_q}(q_1,q_2)_\ell \otimes d \otimes (p_1,p_2)_k \otimes c.$$

If $i+j=-k-\ell$, then $d_p=-k$ and $d_q=-\ell$, and clearly $\phi_{D,C}\circ\tau_1=\tau_2\circ\phi_{C,D}$. If $i+j=-k-\ell-1$, then either $d_p=-k$ and $d_q=-\ell-1$, or $d_p=-k-1$ and $d_q=-\ell$. In both cases, the desired equality holds.

To check that $\phi_{C,D}$ is a pluripotential weak equivalence, note that every complex is non-naturally isomorphic to a sum of complexes of the form:

$$\text{(points)}\quad \cdots\to 0\to \mathbf{k}\to 0\to\cdots \qquad\text{and}\qquad \text{(lines)}\quad \cdots\to 0\to\mathbf{k}\xrightarrow{id}\mathbf{k}\to 0\to\cdots.$$

Let us denote a point by $\bullet$ and a line by $\bullet\to\bullet$. By naturality of ϕ and the fact that Inf and $\otimes$ commute with direct sums, we only need to check that $\phi_{\bullet,\bullet}, \phi_{\bullet,\bullet\to\bullet}, \phi_{\bullet\to\bullet,\bullet}$ and $\phi_{\bullet\to\bullet,\bullet\to\bullet}$ are weak equivalences. Since $\bullet\to\bullet$ is a contractible cochain complex and Inf and $\otimes$ preserve weak equivalences, it follows that $\phi_{\bullet,\bullet\to\bullet}, \phi_{\bullet\to\bullet,\bullet}$ and $\phi_{\bullet\to\bullet,\bullet\to\bullet}$ are homotopically trivial. Denoting by $\bullet^k$ the point concentrated in degree $k\le 0$, observe that $\mathrm{Inf}(\bullet^k)=\llcorner_k$ and

$$\phi_{\bullet^k,\bullet^\ell}=\iota_{k,\ell}:\llcorner_{k+\ell}\to\llcorner_k\otimes\llcorner_\ell,$$

but this is a pluripotential weak equivalence by Proposition 1.4.5. $\square$

We next describe the lax monoidal structure of the functor $\mathcal{B}$ when restricted to third-quadrant bicomplexes.

Theorem 1.6.2. *The functor* $\mathcal{B}\colon \mathrm{BiCo}_{\mathbf{k}}^{\urcorner}\to\mathrm{Ch}_{\mathbf{k}}^{\le 0}$ *is lax symmetric monoidal with structure morphisms given by:*

- $\tilde\phi=\mathrm{id}:\mathbf{k}\to\mathcal{B}(\mathbf{k})$, *and*
- *for all A and B in* $\mathrm{BiCo}_{\mathbf{k}}^{\urcorner}$,

$$\tilde\phi_{A,B}:\mathcal{B}(A)\otimes\mathcal{B}(B)\to\mathcal{B}(A\otimes B)$$

defined on elements $\overline{a}=(a^{-m,-1},a^{-m+1,-2},\dots,a^{-1,-m})$ *in* $\mathcal{B}(A)^{-m}$ *and* $\overline{b}=(b^{-n,-1},\dots,b^{-1,-n})$ *in* $\mathcal{B}(B)^{-n}$, *for* $m,n>1$, *by*

$$\begin{aligned}\tilde\phi_{A,B}^{-m-n}(\overline{a}\otimes\overline{b}) = {}&\\ \sum_{\substack{\alpha+\beta=m\\ \gamma+\delta=n+1\\ \alpha,\beta\ge0,\gamma,\delta\ge1}} &(-1)^{n+1}\frac{\mathrm{L}(m+n,\alpha+\gamma)}{\mathrm{L}(n,\gamma)}\left(\binom{m-1}{\beta-1}\partial a^{-\alpha-1,-\beta}-\binom{m-1}{\alpha-1}\bar\partial a^{-\alpha,-\beta-1}\right)\otimes b^{-\gamma,-\delta}\\ &-\sum_{\substack{\alpha+\beta=m+1\\ \gamma+\delta=n\\ \alpha,\beta\ge1,\gamma,\delta\ge0}}\frac{\mathrm{L}(m+n,\alpha+\gamma)}{\mathrm{L}(m,\alpha)}a^{-\alpha,-\beta}\otimes\left(\binom{n-1}{\delta-1}\partial b^{-\gamma-1,-\delta}-\binom{n-1}{\gamma-1}\bar\partial b^{-\gamma,-\delta-1}\right).\end{aligned}$$

If $\overline{a}\in\mathcal{B}(A)^0$ *or* $\overline{b}\in\mathcal{B}(B)^0$ *we have* $\tilde\phi_{A,B}^{-m-n}(\overline{a}\otimes\overline{b})=\overline{a}\otimes\overline{b}$.

Here, $\mathrm{L}(k,\ell)$ *denote the entries of the Leibniz harmonic triangle [OEI26].*

Proof. Consider a colax monoidal functor

$$F\colon (\mathcal{C},\otimes,I_{\mathcal{C}}) \to (\mathcal{D},\otimes,I_{\mathcal{D}})$$

between monoidal categories, with monoidal structure maps $\nu : F(I_{\mathcal{C}}) \to I_{\mathcal{D}}$ and $\phi_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$. If F has a right adjoint

$$G\colon \mathcal{D} \to \mathcal{C},$$

we can consider the adjoint

$$\tilde{\nu}\colon I_{\mathcal{D}} \to G(I_{\mathcal{C}})$$

of ν, and the natural map

$$\tilde{\phi}\colon G(A) \otimes G(B) \to G(A \otimes B)$$

adjoint to the composite

$$F\left(G(A) \otimes G(B)\right) \xrightarrow{\phi_{G(A),G(B)}} FG(A) \otimes FG(B) \xrightarrow{\varepsilon_A \otimes \varepsilon_B} A \otimes B.$$

The map $\tilde{\phi}$ can equivalently be defined as the composition

$$G(A) \otimes G(B) \xrightarrow{\eta_{G(A)\otimes G(B)}} GF\left(G(A) \otimes G(B)\right) \xrightarrow{G(\phi_{G(A),G(B)})} G\left(FG(A) \otimes FG(B)\right) \xrightarrow{G(\varepsilon_A \otimes \varepsilon_B)} G\left(A \otimes B\right).$$

Here η and ε denote, respectively, the unit and counit of the adjunction. The pair $(\tilde{\nu},\tilde{\phi})$ defines a lax monoidal structure on the right adjoint G. The dual construction is described in [SS03]. If the left adjoint is symmetric colax monoidal, then the right adjoint is symmetric lax monoidal.

The unit and the counit of the adjunction

$$\mathrm{Inf} : \mathrm{Ch}_{\mathbf{k}}^{\leq 0} \rightleftarrows \mathrm{BiCo}_{\mathbf{k}}^{\urcorner} : \mathcal{B}$$

given in the second proof of Theorem 1.5.3 and the colax structure of Inf given in Proposition 1.6.1 yield the explicit description of the lax monoidal structure on $\mathcal{B}$ when restricted to third-quadrant bicomplexes. $\square$

Chapter 2

Operadic algebras

This chapter explores a new context for operadic calculus and Koszul duality arising in the study of bidifferential bigraded algebras up to pluripotential weak equivalence. Namely, we study operads in bicomplexes. We prove the existence and uniqueness of minimal models of operads with respect to pluripotential weak equivalence in two complementary approaches: à la Sullivan, building minimal models by induction on arity, and via Koszul duality. This allows us to define homotopy algebras in this context.

We then study the homotopy theory of operadic algebras in bicomplexes. To this end, we construct the adjoint pair given by the bar and cobar functors. In particular, we prove an equivalence of homotopy categories between algebras over a Koszul operad and algebras over its minimal model.

Finally, we study in detail the case of the associative operad. We introduce the category of pluripotential A_∞-algebras, that is, algebras over the pluripotential minimal model of the associative operad, and establish a Pluripotential Homotopy Transfer Theorem for bidifferential bigraded associative algebras. We then adapt this construction to define the commutative analogue of pluripotential A_∞-algebras and prove the corresponding homotopy transfer theorem for commutative bidifferential bigraded algebras.

As a further step in the associative setting, we study algebras equipped with a compatible pairing. More precisely, we define cyclic A_∞^{pp}-algebras and prove a cyclic homotopy transfer theorem.

Throughout this chapter, $\mathbf{k}$ will denote a field of characteristic zero.

2.1 $\mathbb{S}$-modules

For multiple constructions in operad theory, it is convenient to work with the category $\mathbb{S}$-Mod of $\mathbb{S}$-modules or, more generally, with $\mathbb{S}$-$\mathrm{Ch}^{\leq 0}$, the category of differential graded $\mathbb{S}$-modules. In our context, we will consider bidifferential bigraded $\mathbb{S}$-modules. That is, we will work over the closed symmetric monoidal model category of bicomplexes $\mathrm{BiCo}_{\mathbf{k}}$, although most of the constructions are valid over any symmetric monoidal category. We will review their basic properties and present the Schur functor, a key ingredient to define operadic algebras. Most of the content of this section is a direct adaptation to the bicomplex setting of results presented in [LV12] and [Fre04], except for the extension of the inflation functor to $\mathbb{S}$-modules.

Definition 2.1.1. A *bidifferential bigraded* $\mathbb{S}$*-module* (bb-$\mathbb{S}$-module for short) M is given by a sequence of bicomplexes $\{M(n)\}_{n\in\mathbb{Z}_{\geq 0}}$ together with a right $\mathbb{S}_n$ action, where $\mathbb{S}_n$ denotes the nth symmetric group.

If $\mu \in M(n)^{p,q}$, we say that μ is of *arity* n and bidegree (p,q). We will denote by $|\mu| = p+q$ its total degree.

Definition 2.1.2. A *morphism of* bb-$\mathbb{S}$*-modules* $f\colon M \to N$ is a sequence of equivariant bicomplex morphisms $f_n\colon M(n) \to N(n)$.

We denote the category of bb-$\mathbb{S}$-modules by $\mathbb{S}$-BiCo.

Remark 2.1.3. Note that we can view every bb-$\mathbb{S}$-module M as a family of bicomplexes $\{M(n)\}_{n\in\mathbb{Z}_{\geq 0}}$, where each $M(n)^{p,q}$ is a right $\mathbf{k}[\mathbb{S}_n]$-module for all (p,q).

Let M and N be two bb-$\mathbb{S}$-modules. Define their *tensor product* as

$$(M\otimes N)(n) := \bigoplus_{p+q=n} \mathrm{Ind}_{\mathbb{S}_p\times\mathbb{S}_q}^{\mathbb{S}_n} M(p)\otimes N(q).$$

Here $\mathrm{Ind}_{\mathbb{S}_p\times\mathbb{S}_q}^{\mathbb{S}_n}$ denotes the induced $\mathbb{S}_n$-representation. That is,

$$\mathrm{Ind}_{\mathbb{S}_p\times\mathbb{S}_q}^{\mathbb{S}_n} M(p)\otimes N(q) = (M(p)\otimes N(q)) \underset{\mathbb{S}_p\times\mathbb{S}_q}{\otimes} \mathbf{k}[\mathbb{S}_n].$$

The *composition product* is given by

$$M\circ N = \bigoplus_{n\geq 0} M(n) \underset{\mathbb{S}_n}{\otimes} N^{\otimes n}.$$

Explicitly, the n-ary component of the composition is:

$$(M\circ N)(n) = \bigoplus_{k\geq 0} M(k) \underset{\mathbb{S}_k}{\otimes} \left(\bigoplus_{i_1+\cdots+i_k=n} \mathrm{Ind}_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}}^{\mathbb{S}_n} \left(N(i_1)\otimes\cdots\otimes N(i_k)\right)\right)$$

where the action of $\mathbb{S}_k$ is given by permuting the factors in the tensor product

$$N(i_1)\otimes\cdots\otimes N(i_k)$$

via permutation of the indices $(i_1,\ldots,i_k)$. Therefore, an element of $(M\circ N)(n)$ is a linear combination of tuples

$$(\mu;\mu_1,\ldots,\mu_k)\sigma$$

where $\mu \in M(k)$, $\mu_j \in N(i_j)$, $\sigma\in\mathbb{S}_n$ and $i_1+\cdots+i_k = n$. Such tuples are subject to the following relations:

$$(\mu\tau;\mu_1,\ldots,\mu_k) = (\mu;\mu_{\tau^{-1}(1)},\ldots,\mu_{\tau^{-1}(k)})\tau_{i_1,\ldots,i_k}, \tag{2.1}$$

where $\tau_{i_1,\ldots,i_k}$ is the block permutation of $\{1,\ldots,n\}$ that divides it into k blocks of sizes $i_1,\ldots,i_k$, and permutes these blocks according to τ and

$$(\mu;\mu_1\tau_1,\ldots,\mu_k\tau_k) = (\mu;\mu_{\tau^{-1}(1)},\ldots,\mu_{\tau^{-1}(k)})\tau_1\sqcup\cdots\sqcup\tau_k, \tag{2.2}$$

where each $\tau_j \in \mathbb{S}_{i_j}$, and $\tau_1 \sqcup \cdots \sqcup \tau_k \in \mathbb{S}_n$ is the permutation that acts within each block of size i_j by τ_j.

To a bb-$\mathbb{S}$-module M we may associate an endofunctor in $\mathrm{BiCo}_{\mathbf{k}}$ called the *Schur functor*

$$M : \mathrm{BiCo}_{\mathbf{k}} \to \mathrm{BiCo}_{\mathbf{k}}$$

given by

$$M(A) = \bigoplus_{n \geq 0} M(n) \underset{\mathbb{S}_n}{\otimes} A^{\otimes n}.$$

This functor satisfies

$$(M \otimes N)(-) \cong M(-) \otimes N(-) \text{ and } (M \circ N)(-) \cong M(N(-))$$

(see for instance Section 5.1 of [LV12], or Section 1.2 in [GJ94]). Moreover, we have:

Proposition 2.1.4. *The triple* $(\mathbb{S}\text{-BiCo}, \otimes, \mathbb{1})$ *is a symmetric monoidal category, with* $\mathbb{1} = (\mathbf{k}, 0, \dots)$. *The triple* $(\mathbb{S}\text{-BiCo}, \circ, \mathrm{I})$ *is a monoidal category, with* $\mathrm{I} = (0, \mathbf{k}, 0, \dots)$.

To finish this section, we adapt the inflation functor, defined in Section 1.5, to $\mathbb{S}$-modules of cochain complexes and bicomplexes:

$$\mathrm{Inf}\colon \mathbb{S}\text{-}\mathrm{Ch}_{\mathbf{k}}^{\leq 0} \longrightarrow \mathbb{S}\text{-}\mathrm{BiCo}_{\mathbf{k}}^{\urcorner}.$$

Let $\rho\colon \mathbb{S}_n \to \mathrm{Aut}_{\mathbf{k}}(C)$, be a right $\mathbb{S}_n$ action on a cochain complex C. The inflation functor induces a group morphism

$$\mathrm{Aut}_{\mathbf{k}}(C) \to \mathrm{Aut}_{\mathbf{k}}(\mathrm{Inf}(C))$$

given by

$$\mathrm{Inf}(f)((p,q)_n \otimes c) = (p,q)_n \otimes f(c).$$

By composition, we obtain a group morphism

$$\begin{tikzcd} \mathbb{S}_n \arrow[d, "\rho"'] \arrow[dr, dashed] & \\ \mathrm{Aut}_{\mathbf{k}}(C) \arrow[r] & \mathrm{Aut}_{\mathbf{k}}(\mathrm{Inf}(C)) \end{tikzcd}$$

which defines a right action on $\mathrm{Aut}_{\mathbf{k}}(\mathrm{Inf}(C))$.

Let M be in $\mathbb{S}\text{-}\mathrm{Ch}^{\leq 0}$, then we define

$$\mathrm{Inf}(M)(n) := \mathrm{Inf}(M(n)).$$

The category $(\mathbb{S}\text{-}\mathrm{Ch}^{\leq 0}, \circ, \mathrm{I})$ and category $(\mathbb{S}\text{-}\mathrm{BiCo}^{\urcorner}, \circ, \mathrm{I})$ are monoidal categories. Then,

Proposition 2.1.5. *The functor*

$$\mathrm{Inf}\colon \mathbb{S}\text{-}\mathrm{Ch}^{\leq 0} \to \mathbb{S}\text{-}\mathrm{BiCo}^{\urcorner}$$

is colax monoidal. Moreover, for all M and N in $\mathbb{S}$-$\mathrm{Ch}^{\leq 0}$ the comparison map

$$\mathrm{Inf}(M \circ N) \to \mathrm{Inf}(M) \circ \mathrm{Inf}(N)$$

is a weak equivalence, i.e. an arity-wise pluripotential weak equivalence.

Proof. We will construct natural equivariant maps

$$\psi_{M,N}(n) : \mathrm{Inf}(M \circ N)(n) \to (\mathrm{Inf}(M) \circ \mathrm{Inf}(N))(n)$$

for any dg-$\mathbb{S}$-modules M and N in non-positive degrees, making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Inf}\big((M \circ N) \circ P\big) \arrow[rr, "\cong"] \arrow[d, "\psi_{M\circ N,P}"'] & & \mathrm{Inf}\big(M \circ (N \circ P)\big) \arrow[d, "\psi_{M,N\circ P}"] \\
\mathrm{Inf}(M \circ N) \circ \mathrm{Inf}(P) \arrow[d, "\psi_{M,N}\circ \mathrm{id}"'] & & \mathrm{Inf}(M) \circ \mathrm{Inf}(N \circ P) \arrow[d, "\mathrm{id}\circ\psi_{N,P}"] \\
\big(\mathrm{Inf}(M) \circ \mathrm{Inf}(N)\big) \circ \mathrm{Inf}(P) \arrow[rr, "\cong"] & & \mathrm{Inf}(M) \circ \big(\mathrm{Inf}(N) \circ \mathrm{Inf}(P)\big)
\end{tikzcd}$$

and satisfying the unitality conditions with respect to $\mathrm{id} : \mathrm{Inf}(\mathrm{I}) = \mathrm{I} \to \mathrm{I}$. The horizontal maps in the above square are induced by the symmetric monoidal structures on cochain complexes and bicomplexes, respectively.

First, we have

$$\mathrm{Inf}(M \circ N)(n) = \bigoplus_{s\geq 0} \bigoplus_{\substack{i_1+\cdots+i_k=n \\ s_0+s_1+\cdots+s_k=s}} \mathsf{L}_s M(k)^{s_0} \otimes_{\mathbb{S}_k} \left(\mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}} N(i_1)^{s_1} \otimes \cdots \otimes N(i_k)^{s_k}\right)$$

and

$$(\mathrm{Inf}(M) \circ \mathrm{Inf}(N))(n) =$$
$$\bigoplus_{s\geq 0} \bigoplus_{\substack{i_1+\cdots+i_k=n \\ s_0+s_1+\cdots+s_k=s}} \mathsf{L}_{s_0} M(k)^{s_0} \otimes_{\mathbb{S}_k} \left(\mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}} \mathsf{L}_{s_1} N(i_1)^{s_1} \otimes \cdots \otimes \mathsf{L}_{s_k} N(i_k)^{s_k}\right).$$

To compare these two expressions, we use the maps

$$\iota_{s_0,s_1,\ldots,s_k} : \mathsf{L}_s \to \mathsf{L}_{s_0} \otimes \mathsf{L}_{s_1} \otimes \cdots \otimes \mathsf{L}_{s_k}$$

defined in Section 1.4, which are given by

$$(p,q)_r \mapsto \sum_{\substack{p_0+\cdots+p_k=p \\ q_0+\cdots+q_k=q}} (-1)^{\sum_i (d_0+\cdots+d_{i-1})(d_i+r_i)} (p_0,q_0)_{s_0} \otimes \cdots \otimes (p_k,q_k)_{s_k}$$

Elements of $\mathrm{Inf}\big((M \circ N)\big)(n)$ are linear combinations of formal composites

$$(p,q)_s \otimes (\nu_0; \nu_1, \ldots, \nu_k)\sigma$$

where $s =\mid \nu_0 \mid + \cdots + \mid \nu_k \mid$, $\sigma \in \mathbb{S}_n$, $\nu_0 \in M(k)$ and $\nu_1 \in N(i_1), \ldots, \nu_k \in N(i_k)$ with

$i_1 + \cdots + i_k = n$. With this description we define the comparison map

$$\psi_{M,N} : \operatorname{Inf}(M \circ N) \to \operatorname{Inf}(M) \circ \operatorname{Inf}(N)$$

by letting

$$\psi_{M,N}(n)((p,q)_s \otimes (\nu_0;\nu_1,\ldots,\nu_k)\sigma) := \sum_{\substack{p_0+\cdots+p_k=p\\ q_0+\cdots+q_k=q}} (-1)^{\varepsilon}(\nu_0^{p_0,q_0};\nu_1^{p_1,q_1},\ldots,\nu_k^{p_k,q_k})\sigma$$

where $\varepsilon = \sum_i (d_0 + \cdots + d_{i-1})(d_i + r_i)$ and $\nu_r^{p_r,q_r} := (p_r,q_r)_{d_r} \otimes \nu_r$ and $d_r = |\nu_r|$. The map $\psi(n)$ is obviously equivariant.

The fact that it is a map of bicomplexes and the required properties of $\psi_{M,N}$ are now directly deduced from the case of chain complexes of **k**-modules of Proposition 1.6.1. Namely, the above diagram commutes and $\psi_{M,N}$ is a weak equivalence. □

Proposition 2.1.6. *The inflation functor*

$$\operatorname{Inf}\colon \mathbb{S}\text{-}\mathrm{Ch}^{\leq 0} \to \mathbb{S}\text{-}\mathrm{BiCo}^{\urcorner}$$

sends quasi-isomorphisms to pluripotential weak equivalences.

Proof. Let $\varphi\colon M \to N$ be a quasi-isomorphism in $\mathbb{S}\text{-}\mathrm{Ch}^{\leq 0}$, i.e. a sequence of quasi-isomorphisms of cochain complexes $\varphi(n)\colon M(n) \to N(n)$ compatible with the $\mathbb{S}_n$ action. Then,

$$\operatorname{Inf}(\varphi)\colon \operatorname{Inf}(M) \to \operatorname{Inf}(N)$$

is a sequence of maps of bicomplexes

$$\operatorname{Inf}(\varphi)(n)\colon \operatorname{Inf}(M(n)) \to \operatorname{Inf}(N(n))$$

compatible with the action of $\mathbb{S}_n$. By Proposition 1.5.7, the functor $\operatorname{Inf}\colon \mathrm{Ch}^{\leq 0} \to \mathrm{BiCo}^{\urcorner}_{\mathbf{k}}$ sends quasi-isomorphisms to pluripotential weak equivalences, we obtain the desired result. □

There is a similar story in the non-symmetric setting. In this case, one considers sequences $M = \{M(n)\}_{n\geq 0}$ of bicomplexes without the $\mathbb{S}_n$-action. The associated functor is defined by

$$M(A) = \bigoplus_{n\geq 0} M(n) \otimes A^{\otimes n}.$$

All the constructions presented above admit an obvious non-symmetric analog.

2.2 Operads and operadic algebras

In this section, we recall the definitions of operads and operadic algebras along with some important results, adapted to the pluripotential setting when needed. Most of the material presented here is taken from [GJ94], [LV12], and [Fre04].

Definition 2.2.1. An *operad* is a monoid in the monoidal category $(\mathbb{S}\text{-}\mathrm{BiCo}, \circ, \mathrm{I})$, that is, a bb-$\mathbb{S}$-module $\mathcal{P}$ together with morphisms

$$\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P} \quad \text{and} \quad \eta : \mathrm{I} \to \mathcal{P}$$

called the *composition* and *unit* maps, respectively, satisfying associativity

$$\begin{tikzcd}
(\mathcal{P} \circ \mathcal{P}) \circ \mathcal{P} \arrow[r, "\cong"] \arrow[d, "\gamma\circ\mathrm{id}"'] & \mathcal{P} \circ (\mathcal{P} \circ \mathcal{P}) \arrow[r, "\mathrm{id}\circ\gamma"] & \mathcal{P} \circ \mathcal{P} \arrow[d, "\gamma"] \\
\mathcal{P} \circ \mathcal{P} \arrow[rr, "\gamma"] & & \mathcal{P}
\end{tikzcd}$$

and unitality

$$\begin{tikzcd}
\mathrm{I} \circ \mathcal{P} \arrow[r, "\eta\circ\mathrm{id}"] \arrow[dr, "="'] & \mathcal{P} \circ \mathcal{P} \arrow[d, "\gamma"] & \mathcal{P} \circ \mathrm{I} \arrow[l, "\mathrm{id}\circ\eta"'] \arrow[dl, "="] \\
& \mathcal{P} &
\end{tikzcd}$$

conditions. We denote by id the image of $1_{\mathbf{k}}$ under η.

In particular, the operad structure is determined by maps

$$\mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_k) \to \mathcal{P}(i_1 + \cdots + i_k)$$

subject to associativity, unitality and equivariance conditions. Moreover, the differentials of $\mathcal{P}$, ∂ and $\bar{\partial}$, are *derivations* for γ

$$\begin{aligned}\partial(\gamma(\mu; \mu_1, \ldots, \mu_k)) &:= \gamma(\partial(\mu); \mu_1, \ldots, \mu_k) \\ &+ \sum_{i=1}^{k} (-1)^{\epsilon_i} \gamma(\mu; \mu_1, \ldots, \partial(\mu_i), \ldots, \mu_k),\end{aligned}$$

where $\epsilon_i = |\mu| + |\mu_1| + \cdots + |\mu_{i-1}|$. The same identity holds for $\bar{\partial}$.

Let $\mathcal{P}$ be an operad and let $\mu \in \mathcal{P}(m)$ an element in arity m and $\nu \in \mathcal{P}(n)$ and element in arity n. We will sometimes use the *partial composition*

$$(\mu, \nu) \longmapsto \mu \circ_i \nu \in \mathcal{P}(m - 1 + n)$$

defined for $1 \leq i \leq m$ by:

$$- \circ_i - \; : \; \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m - 1 + n), \qquad \mu \circ_i \nu \; := \; \gamma(\mu; \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{i-1}, \nu, \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{m-i}).$$

Definition 2.2.2. A *morphism of operads* $f : \mathcal{P} \to \mathcal{Q}$ is a morphism of bb-$\mathbb{S}$-modules which is compatible with the composition and unit maps.

We denote the category of operads in $\mathrm{BiCo}_{\mathbf{k}}$ by Op(BiCo).

Example 2.2.3. Let A be a bicomplex. Consider the internal hom defined in Section 1.1

$$\underline{\mathrm{Hom}}(A, B)^{p,q} = \prod \mathrm{Hom}_{\mathbf{k}}(A^{i,j}, B^{i+p,j+q}),$$

with differentials

$$\partial f := [\partial, f] \quad \text{and} \quad \bar{\partial} f := [\bar{\partial}, f].$$

Define the following bb-$\mathbb{S}$-module

$$\mathcal{E}nd(n) := \underline{\mathrm{Hom}}(A^{\otimes n}, A)$$

where the action of $\mathbb{S}_n$ is induced by the left $\mathbb{S}_n$ action on $A^{\otimes n}$. We can endow it with an operadic structure given by composition of endomorphisms

$$(f; f_1, \ldots, f_k) := f(f_1 \otimes \cdots \otimes f_k).$$

The resulting operad is called the *endomorphism* operad.

Example 2.2.4. The *associative* operad is given by

$$Ass(n) = \mathbf{k}[\mathbb{S}_n]$$

with composition law given by $e_k \circ_i e_\ell \mapsto e_{k+\ell-1}$, where e_k denotes the trivial permutation on k elements. One can define the associative operad in the non-symmetric setting by

$$As(n) = \mathbf{k}\langle \mu_n \rangle$$

with composition law given by $\mu_k \circ_i \mu_\ell = \mu_{k+\ell-1}$.

Example 2.2.5. The *commutative operad* is given by

$$Com(n) = \mathbf{k}$$

with trivial $\mathbb{S}_n$ action. The composition is given as in the associative operad.

If $\mathcal{P}$ is an operad, then Proposition 2.1.4 shows that the Schur functor

$$\mathcal{P} : \mathrm{BiCo}_{\mathbf{k}} \to \mathrm{BiCo}_{\mathbf{k}}$$

defines a monad.

Definition 2.2.6. A *$\mathcal{P}$-algebra* is an algebra over the monad $\mathcal{P}$, that is, an object A in $\mathrm{BiCo}_{\mathbf{k}}$ together with maps of bicomplexes $\gamma_A \colon \mathcal{P}(A) \to A$ and $\eta_A \colon A \to \mathcal{P}(A)$ such that the following diagrams commute:

$$\begin{tikzcd} \mathcal{P} \circ \mathcal{P}(A) \arrow[r, "\gamma(A)"] \arrow[d, "\mathcal{P}(\gamma_A)"'] & \mathcal{P}(A) \arrow[d, "\gamma_A"] \\ \mathcal{P}(A) \arrow[r, "\gamma_A"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & \mathcal{P}(A) \arrow[d, "\gamma_A"] \\ & A \end{tikzcd}$$

The $\mathcal{P}$-algebra structure is determined by maps

$$\gamma_n : \mathcal{P}(n) \underset{\mathbb{S}_n}{\otimes} A^{\otimes n} \to A.$$

One should think of $\mathcal{P}(n)$ as the space that parametrizes all the possible ways of combining n elements of the $\mathcal{P}$-algebra.

Using the fact that there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{S}_n}(\mathcal{P}(n), \mathrm{Hom}(A^{\otimes n}, A)) \cong \mathrm{Hom}(\mathcal{P}(n) \underset{\mathbb{S}_n}{\otimes} A^{\otimes n}, A),$$

one can alternatively define an algebra over an operad $\mathcal{P}$ as a morphism of operads

$$\mathcal{P} \to \mathcal{E}nd(A).$$

The image of an element $\mu \in \mathcal{P}(n)$ is an operation $\mu_n : A^{\otimes n} \to A$. We denote the category of $\mathcal{P}$-algebras as $\mathcal{P}$-Alg.

Example 2.2.7. Algebras over the associative operad are *bidifferential bigraded algebras* (bba's). That is, a bicomplex $(A, \partial, \bar{\partial})$ equipped with a bidegree-preserving associative product μ such that

$$\partial\mu = \mu(\partial \otimes \mathrm{id}) + \mu(\mathrm{id} \otimes \partial), \qquad \bar{\partial}\mu = \mu(\bar{\partial} \otimes \mathrm{id}) + \mu(\mathrm{id} \otimes \bar{\partial}).$$

Algebras over the commutative operad are *commutative bidifferential bigraded algebras* (cbba's), that is, bba's whose associative product is, in addition, commutative.

The complexified de Rham algebra of a complex manifold is an example of a cbba (see Chapter 3).

The following definition will be used to twist differentials on a $\mathcal{P}$-algebra.

Definition 2.2.8. A *derivation* of a $\mathcal{P}$-algebra A is a linear map $d : A \to A$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{\gamma_A} & A \\ {\scriptstyle \mathrm{id}\circ' d}\downarrow & & \downarrow{\scriptstyle d} \\ \mathcal{P}(A) & \xrightarrow{\gamma_A} & A \end{array}$$

where $\mathrm{id} \circ' d$ denotes the map given by $\sum_{i=1}^{n} \mathrm{id}^i \otimes d \otimes \mathrm{id}^{n-i}$ in $\mathcal{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n}$.

If A is a $\mathcal{P}$-algebra, the bigraded vector space of derivations of A, $\mathrm{Der}(A)$, is a bidifferential bigraded Lie algebra, with bracket the graded commutator and differentials the graded commutators with the differentials of A.

Definition 2.2.9. A *pair of differentials* of a $\mathcal{P}$-algebra consists in two derivations $(\tau, \overline{\tau})$ of degree $(1, 0)$ and $(0, 1)$, respectively, such that

$$(\partial + \tau)^2 = 0, \quad (\bar{\partial} + \overline{\tau})^2 = 0 \quad \text{and} \quad (\partial + \tau)(\bar{\partial} + \overline{\tau}) + (\bar{\partial} + \overline{\tau})(\partial + \tau) = 0.$$

If $(\tau, \overline{\tau})$ is a pair of differentials, then the bicomplex $(A, \partial + \tau, \bar{\partial} + \overline{\tau})$ is a $\mathcal{P}$-algebra with the same structure map $\gamma_A \colon \mathcal{P}(A) \to A$ as A.

Since an operad $\mathcal{P}$ is a monad, we have the definition of a free $\mathcal{P}$-algebra. The *free $\mathcal{P}$-algebra* over a bicomplex A is the algebra given by $\mathcal{P}(A)$ itself, and the algebra structures $\gamma_A : \mathcal{P} \circ \mathcal{P}(A) \to \mathcal{P}(A)$ and $\eta_A : \mathcal{P}(A) \to \mathcal{P} \circ \mathcal{P}(A)$ are given by the monad structure of $\mathcal{P}$.

For any $\mathcal{P}$-algebra A and any linear map $f : V \to A$, there is a unique $\mathcal{P}$-algebra extension $\tilde{f} : \mathcal{P}(V) \to A$ of f:

$$\begin{tikzcd} V \arrow[r, "\eta"] \arrow[dr, "f"'] & \mathcal{P}(V) \arrow[d, "\tilde{f}"] \\ & A. \end{tikzcd}$$

Note that a free algebra is unique up to a unique isomorphism.

The following proposition is very useful for constructing derivations on free algebras, see for instance Section 6.3.6 in [LV12], [GJ94], among others.

Proposition 2.2.10. *Any derivation d on a free $\mathcal{P}$-algebra $\mathcal{P}(A)$ is uniquely determined by its restriction to the generators, that is, by a linear map $A \to \mathcal{P}(A)$.*

More explicitly, given a map $\varphi : A \to \mathcal{P}(A)$, the unique derivation d_φ of the free $\mathcal{P}$-algebra $\mathcal{P}(A)$ extending φ is defined by

$$d_\varphi = (\gamma_{(1)} \circ \mathrm{id}_A)\,(\mathrm{id}_{\mathcal{P}} \circ' \varphi).$$

An algebra is said to be *quasi-free* if when we forget the differentials it is a free algebra.

The *free operad* over a bb-$\mathbb{S}$-module M is an operad $\mathbb{T}(M)$ with a bb-$\mathbb{S}$-module morphism $\eta(M) : M \to \mathbb{T}(M)$ such that given an operad $\mathcal{P}$, any map $f : M \to \mathcal{P}$ of bb-$\mathbb{S}$-modules, extends uniquely into an operad morphism $\tilde{f} : \mathbb{T}(M) \to \mathcal{P}$:

$$\begin{tikzcd} M \arrow[r, "\eta(M)"] \arrow[dr, "f"'] & \mathbb{T}(M) \arrow[d, "\tilde{f}"] \\ & \mathcal{P} \end{tikzcd}$$

The functor $\mathbb{T}\colon \mathbb{S}\text{-BiCo} \to \mathrm{Op(BiCo)}$ is the left adjoint to the forgetful functor $U\colon \mathrm{Op(BiCo)} \to \mathbb{S}\text{-BiCo}$. Arity-wise it can be computed as

$$\mathbb{T}(M)(n) := \operatorname*{colim}_{t \in \mathrm{T}(n)} M(t),$$

where $\mathrm{T}(n)$ denotes the set of rooted trees with n leaves. For a tree t, the object $M(t)$ is the *tree-wise tensor product* of the $\mathbb{S}$-module M along the vertices of t:

$$M(t) = \bigotimes_{v \in V(t)} M(\mathrm{in}(v)),$$

where $\mathrm{in}(v)$ denotes the number of inputs of the vertex v. For details, see Section 5.6.1 of [LV12] or [GJ94].

Remark 2.2.11. When $M(0) = 0$, $\mathbb{T}(M)(n) = \bigoplus_{t \in \mathrm{T}(n)} M(t)$.

It is helpful to think of an element of $\mathbb{T}(M)(n)$ as a sum of rooted trees in which each vertex v is decorated by an element of $M(\mathrm{in}(v))$ and each leaf is decorated by an element of $\{1, \dots, n\}$. The operad structure of the $\mathbb{S}$-module $\mathbb{T}(M)$ is given by grafting of trees.

The following property will be very useful.

Proposition 2.2.12. *Let M be a* bb-$\mathbb{S}$*-module. Any derivation $d_\phi : \mathbb{T}(M) \to \mathbb{T}(M)$ of the free operad $\mathbb{T}(M)$ is completely characterized by the image on the generators: $\phi : M \to \mathbb{T}(M)$.*

When representing an element of $\mathbb{T}(M)$ by a labeled tree, its image under d_ϕ is the sum of labeled trees, where ϕ has been applied once and only once to every vertex. Such operads are called *quasi-free* operads. It means that the underlying bigraded operad, forgetting the differentials, is free.

Remark 2.2.13. Let M be a bb-$\mathbb{S}$-module, the free operad on M, $\mathbb{T}(M)$, has a weight-grading. By definition, the *weight* $w(\mu)$ of an operation $\mu \in \mathbb{T}(M)$ is defined as follows:

$$w(\mathrm{id}) = 0, \qquad w(\mu) = 1 \text{ when } \mu \in M,$$

and

$$w(\nu;\, \nu_1, \ldots, \nu_n) = w(\nu) + w(\nu_1) + \cdots + w(\nu_n).$$

We denote by $\mathbb{T}(M)^{(r)}$ the bb-$\mathbb{S}$-module of operations of weight r.

A completely analogous construction exists for the free non-symmetric operad using planar trees instead, see [LV12, Sections 5.9.5 and 5.9.6] for details.

2.3 Cooperads and coalgebras over cooperads

In this section we review the dual notions of operads and algebras, that is, cooperads and coalgebras.

Definition 2.3.1. A *cooperad* is a comonoid in the monoidal category $(\mathbb{S}\text{-BiCo}, \circ, \mathrm{I})$. Explicitly, this consists of a bb-$\mathbb{S}$-module $\mathcal{C}$ equipped with morphisms

$$\Delta : \mathcal{C} \to \mathcal{C} \circ \mathcal{C} \quad \text{ and } \quad \varepsilon : \mathcal{C} \to \mathrm{I},$$

satisfying coassociativity and counit axioms

$$\begin{tikzcd}
\mathcal{C} \arrow[rrr, "\Delta"] \arrow[d, "\Delta"'] & & & \mathcal{C} \circ \mathcal{C} \arrow[d, "\mathrm{id}\circ\Delta"] \\
\mathcal{C} \circ \mathcal{C} \arrow[r, "\Delta\circ\mathrm{id}"] & (\mathcal{C} \circ \mathcal{C}) \circ \mathcal{C} \arrow[rr, "\cong"] & & \mathcal{C} \circ (\mathcal{C} \circ \mathcal{C})
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{C} \arrow[dl, "="'] \arrow[d, "\Delta"] \arrow[dr, "="] & \\
\mathrm{I} \circ \mathcal{C} & \mathcal{C} \circ \mathcal{C} \arrow[l, "\varepsilon\circ\mathrm{id}"'] \arrow[r, "\mathrm{id}\circ\varepsilon"] & \mathcal{C} \circ \mathrm{I}
\end{tikzcd}$$

We denote the category of cooperads by CoOp(BiCo). We will sometimes adopt the following notation

$$\Delta(\mu) = \sum (\nu;\, \nu_1, \ldots, \nu_k)\sigma.$$

And sometimes use the shorter notation

$$\Delta(\nu) = \sum \nu_{(1)} \circ \nu_{(2)},$$

where

$$\nu_{(2)} = (\nu_1, \ldots, \nu_k)\sigma \in \mathcal{C}^{\otimes k}(n).$$

The differentials $\partial_{\mathcal{C}}$ and $\bar{\partial}_{\mathcal{C}}$ are coderivations, under this notation, this means

$$\Delta\partial_{\mathcal{C}}(c) = \sum(\partial_{\mathcal{C}}(c)\,;\, c_1, \ldots, c_k)\sigma + \sum\sum_{i=1}^{k}(-1)^{\theta_i}\,(c\,;\, c_1, \ldots, \partial_{\mathcal{C}}(c_i), \ldots, c_k)\sigma,$$

where $\theta_i = |c| + |c_1| + \cdots + |c_{i-1}|$.

Given a cooperad $(\mathcal{C}, \Delta, \eta)$, one may isolate from the full decomposition map $\Delta : \mathcal{C} \to \mathcal{C} \circ \mathcal{C}$ the part that corresponds to decomposing into two parts. Let M and N be two bb-$\mathbb{S}$-modules, we denote by $M \circ_{(1)} N$ the bb-$\mathbb{S}$-module defined in arity n by

$$(M \circ_{(1)} N)(n) \;=\; \bigoplus_{k+\ell=n+1} M(k) \otimes_{\mathbb{S}_k} \left(\bigoplus_{1\leq j\leq k} \mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_1^{\times(j-1)}\times\mathbb{S}_\ell\times\mathbb{S}_1^{\times(k-j)}} \mathrm{I}^{\otimes j-1} \otimes N(\ell) \otimes \mathrm{I}^{k-j} \right).$$

This yields the *infinitesimal decomposition map* of $\mathcal{C}$, defined as the composition

$$\Delta_{(1)} : \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C} \;\longrightarrow\; \mathcal{C} \circ_{(1)} \mathcal{C}.$$

The infinitesimal decomposition may be written in the form

$$\Delta_{(1)}(\nu) = \sum_i (\nu_i' \circ_{e_i} \nu_i'')\,\tau_i,$$

for $\nu_i' \in \mathcal{C}(k_i')$, $\nu_i'' \in \mathcal{C}(k_i'')$, and $\tau_i \in \mathbb{S}_n$, where

$$\nu_i' \circ_{e_i} \nu_i'' = \nu_i' \circ (\mathrm{id}^{e_i-1} \otimes \nu_i'' \otimes \mathrm{id}^{k_i'-e_i}).$$

If $\mathcal{C}$ is a cooperad, then the Schur functor is a comonad, see [GJ94].

Definition 2.3.2. Let $\mathcal{C}$ be an operad. A *$\mathcal{C}$-coalgebra* is a coalgebra over the comonad $\mathcal{C}$. That is, an object C in $\mathrm{BiCo}_{\mathbf{k}}$ together with maps of bicomplexes $\Delta_C : C \to \mathcal{C}(C)$ and $\varepsilon : \mathcal{C}(C) \to C$ satisfying the coassociativity and counit constraints:

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & \mathcal{C}(C) \arrow[d, "\mathcal{C}(\Delta_C)"] \\ \mathcal{C}(C) \arrow[r, "\Delta_{\mathcal{C}(C)}"] & \mathcal{C} \circ \mathcal{C}(C) \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[dr, equal] & \mathcal{C}(C) \arrow[d, "\varepsilon"] \\ & C \end{tikzcd}$$

The following definition will be used to twist codifferentials on a $\mathcal{C}$-coalgebra.

Definition 2.3.3. A *coderivation* of a $\mathcal{C}$-coalgebra is a linear map $d\colon C \to C$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "d"'] & \mathcal{C}(C) \arrow[d, "\mathrm{id}\circ' d"] \\ C \arrow[r, "\Delta_C"] & \mathcal{C}(C) \end{tikzcd}$$

The bigraded vector space of coderivations on a $\mathcal{C}$-coalgebra C, $\mathrm{Coder}(C)$, is a bidifferential bigraded Lie algebra with bracket the graded commutator and differentials the graded commutators with the differentials of C.

Definition 2.3.4. A *pair of codifferentials* of a $\mathcal{C}$-coalgebra consists in two coderivations $(\tau, \overline{\tau})$ of degree $(1,0)$ and $(0,1)$, respectively, such that

$$(\partial + \tau)^2 = 0, \quad (\bar{\partial} + \overline{\tau})^2 = 0 \quad \text{and} \quad (\partial + \tau)(\bar{\partial} + \overline{\tau}) + (\bar{\partial} + \overline{\tau})(\partial + \tau) = 0.$$

If $(C, \partial, \bar{\partial})$ is a $\mathcal{C}$-coalgebra and $(\tau, \overline{\tau})$ is a pair of codifferentials, then the bicomplex $(C, \partial + \tau, \bar{\partial} + \overline{\tau})$ is a $\mathcal{C}$-coalgebra with the same structure map $\Delta_C \colon C \to \mathcal{C}(C)$ as C.

Since a cooperad $\mathcal{C}$ is a comonad, there is the notion of cofree coalgebra. The *cofree $\mathcal{C}$-coalgebra* over a bicomplex V is the algebra given by $\mathcal{C}(V)$ itself, and the coalgebra structures $\Delta_C \colon \mathcal{C}(V) \to \mathcal{C} \circ \mathcal{C}(V)$ and $\eta_C \colon \mathcal{C} \circ \mathcal{C}(V) \to \mathcal{C}(V)$ are given by the comonad structure of $\mathcal{C}$.

For any $\mathcal{C}$-coalgebra C and any linear map $f \colon C \to V$, there is a unique $\mathcal{C}$-coalgebra extension $\tilde{f} \colon C \to \mathcal{C}(V)$ of f:

$$\begin{tikzcd} & \mathcal{C}(V) \arrow[d, "\varepsilon"] \\ C \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & V \end{tikzcd}$$

Note that a cofree coalgebra is unique up to a unique isomorphism.

The following will be very useful to construct coderivations of cofree coalgebras.

Proposition 2.3.5. *Any coderivation d on a cofree $\mathcal{C}$-coalgebra $\mathcal{C}(V)$ is completely characterized by its projection onto the space of cogenerators V. Explicitly, given a map $\varphi \colon \mathcal{C}(V) \to V$, the unique coderivation d_φ on the cofree $\mathcal{C}$-coalgebra $\mathcal{C}(V)$ that extends φ is given by*

$$d_\varphi \;=\; \big(\mathrm{id}_{\mathcal{C}} \circ (\mathrm{id}_V \,;\, \varphi)\big)\big(\Delta_{(1)} \circ \mathrm{id}_V\big)$$

where $\Delta_{(1)} : \mathcal{C} \to \mathcal{C} \circ_{(1)} \mathcal{C}$ is the infinitesimal decomposition map of the cooperad. The map

$$\mathrm{id}_{\mathcal{C}} \circ (\mathrm{id}_V \,;\, \varphi) : (\mathcal{C} \circ_{(1)} \mathcal{C})(V) \longrightarrow \mathcal{C}(V)$$

acts by applying id_V to the V-tensor factor and applying φ to the $\mathcal{C}(V)$-tensor factor.

The *cofree cooperad* over a bb-$\mathbb{S}$-module M is a cooperad $\mathbb{T}^c(M)$ such that for any bb-$\mathbb{S}$-module morphism $\varphi : \mathcal{C} \to M$ sending id to 0, there exists a unique cooperad morphism $\widetilde{\varphi} : \mathcal{C} \to \mathbb{T}^c(M)$ making the following diagram commute:

$$\begin{tikzcd} C \arrow[r, "\widetilde{\varphi}"] \arrow[dr, "\varphi"'] & F^c(M) \arrow[d] \\ & M \end{tikzcd}$$

The following property will be very useful to construct coderivations

Proposition 2.3.6. *Let M be a* bb-$\mathbb{S}$*-module. Any coderivation $d_\phi \colon \mathbb{T}(M) \to \mathbb{T}(M)$ of the free operad $\mathbb{T}(M)$ is completely characterized by the image to the cogenerators $\phi \colon \mathbb{T}(M) \to M$.*

2.4 Minimal models of operads

We prove the existence and uniqueness of minimal models for reduced operads in $\mathrm{BiCo}_{\mathbf{k}}$ with respect to pluripotential weak equivalences. The models we give are defined inductively on arity, in the spirit of Sullivan's minimal models and following the approach of Markl–Shnider–Stasheff [MSS02].

We begin by introducing principal extensions of operads in bicomplexes and showing that these satisfy a lifting property with respect to pluripotential weak equivalences. We mainly adapt results from [MSS02], [Roi22], and [CR19] on basic homotopy theory of operads to the pluripotential case. In particular, we will introduce a functorial path for operads in bicomplexes and study the associated notion of homotopy equivalence.

We will work with *reduced* operads, i.e., operads $\mathcal{P}$ satisfying $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \mathbf{k}$. Reduced operads are *augmented*, that is, there exists an operad morphism $\varepsilon : \mathcal{P} \to \mathrm{I}$.

Definition 2.4.1. Let $n \geq 2$ be an integer. Let $(\mathcal{P}, \partial_{\mathcal{P}}, \bar{\partial}_{\mathcal{P}})$ be a quasi-free operad, $\mathcal{P} = \mathbb{T}(M)$ where M is a bb-$\mathbb{S}$-mod with $M(0) = M(1) = 0$. An *arity n principal extension* of $\mathcal{P}$ is the quasi-free operad

$$\mathcal{P} \sqcup_d \mathbb{T}(E) := \mathbb{T}(M \oplus E),$$

where E is a bigraded $\mathbb{S}$-module concentrated in arity n and $d = (\partial + \bar{\partial}) : E \to \ker((\partial_{\mathcal{P}} + \bar{\partial}_{\mathcal{P}})(n))$ a map of degree $(1,0)+(0,1)$. The differentials on $\mathcal{P} \sqcup_d \mathbb{T}(E)$ are determined by the differentials of $\mathcal{P}$, d and the Leibniz rule.

Lemma 2.4.2. *Let $\mathcal{P} \sqcup_d \mathbb{T}(E)$ be an arity n principal extension of a quasi-free operad $\mathcal{P} = \mathbb{T}(M)$ and let $\varphi : \mathcal{P} \to \mathcal{Q}$ be a morphism of operads. A morphism $\varphi' \colon \mathcal{P} \sqcup_d \mathbb{T}(E) \to Q$ extending φ is uniquely determined by a morphism of $\mathbb{S}$-modules $f \colon E \to Q(n)$ satisfying $(\partial_{\mathcal{Q}} + \bar{\partial}_{\mathcal{Q}})f = \varphi d$.*

Proof. By the universal property of free operads, any such φ' is determined by its restriction $\varphi'_{|M \oplus E}$, and $\varphi'_{|M} = \varphi$, set $f := \varphi'_{|E}$. We have $(\partial_{\mathcal{Q}} + \bar{\partial}_{\mathcal{Q}})f = (\partial_{\mathcal{Q}} + \bar{\partial}_{\mathcal{Q}})\varphi'_{|E} = \varphi' \circ d_{|E} = \varphi d$. $\square$

The following is a pluripotential version of Lemma 3.139 in [MSS02]. We adapt the proof given in [CR19]. In our current setting, weak equivalences and surjections are given by arity-wise pluripotential weak equivalences and surjections respectively.

Lemma 2.4.3. *Let $\iota \colon \mathcal{P} \to \mathcal{P} \sqcup_d \mathbb{T}(E)$ an arity n principal extension and*

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\varphi"] \arrow[d, "\iota"'] & \mathcal{Q} \arrow[d, two heads, "\omega", "\wr"'] \\ \mathcal{P} \sqcup_d \mathbb{T}(E) \arrow[r, "\psi"] \arrow[ur, dashed, "\psi'"] & \mathcal{R} \end{tikzcd}$$

a commutative diagram of operad morphisms, where ω is a surjective weak equivalence. Then, there exists an operad morphism ψ' making both triangles commute.

Proof. Consider the diagram of bigraded $\mathbf{k}$-vector spaces:

$$\begin{tikzcd} & Z(C(\mathrm{id}_{\mathcal{Q}}(n))) \arrow[d, two heads, "Id^{\oplus 3} \oplus \omega(n)"] \\ E \arrow[r, "\lambda"] \arrow[ur, dashed, "\mu"] & Z(C(\omega(n))) \end{tikzcd}$$

where $Z(\_)$ denotes the space of Bott-Chern cocycles, $C(\_)$ denotes the cone of a map of bicomplexes defined in Section 1.1, and $\lambda = (\varphi \circ \partial\bar{\partial}, -\varphi \circ \partial, -\varphi \circ \bar{\partial}, \psi_{|E})$. We first show this diagram is well defined. Recall that

$$C(\mathrm{id}_{\mathcal{Q}}(n)) = \mathcal{Q}(n)[-1,-1] \oplus \mathcal{Q}(n)[-1,0] \oplus \mathcal{Q}(n)[0,-1] \oplus \mathcal{Q}(n)$$

with differentials given by

$$\begin{aligned} \partial(a, a', a'', b) &= (\partial a, -\partial a', -\partial a'' - a, \partial b + a') \\ \bar{\partial}(a, a', a'', b) &= (\bar{\partial} a, -\bar{\partial} a' + a, -\bar{\partial} a'', \bar{\partial} b + a''). \end{aligned}$$

In the same way,

$$C(\omega(n)) = \mathcal{Q}(n)[-1,-1] \oplus \mathcal{Q}(n)[-1,0] \oplus \mathcal{Q}(n)[0,-1] \oplus \mathcal{R}(n)$$

with differentials given by

$$\begin{aligned} \partial(a, a', a'', b) &= (\partial a, -\partial a', -\partial a'' - a, \partial b + \omega(a')), \\ \bar{\partial}(a, a', a'', b) &= (\bar{\partial} a, -\bar{\partial} a' + a, -\bar{\partial} a'', \bar{\partial} b + \omega(a'')). \end{aligned}$$

Let $(a, a', a'', b) \in Z(C(\mathrm{id}_{\mathcal{Q}}(n)))$, then $(\partial+\bar{\partial})(a) = 0$, $\bar{\partial}(a') = -\partial(a'') = a$, $-(\partial+\bar{\partial})b = a'+a''$, and

$$(\mathrm{id}^{\oplus 3} \oplus \omega(n))(a, a', a'', b) = (a, a', a'', \omega(b)) \in Z(C(\omega(n)))$$

because $-(\partial + \bar{\partial})\omega(b) = -\omega((\partial + \bar{\partial})b) = \omega(a' + a'')$. It is also clear that $\mathrm{Im}(\lambda) \subseteq Z(C(\omega(n)))$ by Lemma 2.4.2.

We now construct a map $\mu : E \to Z(C(\mathrm{id}_{\mathcal{Q}}(n)))$. Since ω is a weak equivalence, we have that $H_{\mathcal{BC}}(C(\omega(n))) = 0$ (see Theorem 1.21 [Ste25a]). Let $(a, a', a'', c) \in Z(C(\omega(n)))$, then there is an element (e, e', e'', f) such that $\partial\bar{\partial}(e, e', e'', f) = (a, a', a'', c)$, note that

$$\partial\bar{\partial}(e, e', e'', f) = (\partial\bar{\partial} e, \partial\bar{\partial} e' - \partial e, \partial\bar{\partial} e'' - \bar{\partial} e, \partial\bar{\partial} f + \omega(-\bar{\partial} e' + \partial e'' + e)).$$

Since $\omega(n)$ is surjective, we may choose $f' \in Q(n)$ such that $\omega(f') = f$. Consider the element $(a, a', a'', \partial\bar{\partial} f' - \bar{\partial} e' + \partial e'' + e) \in Z(\mathrm{id}_{\mathcal{Q}}(n))$. It satisfies

$$(\mathrm{id}^{\oplus 3} \oplus \omega)(a, a', a'', \partial\bar{\partial} f' - \bar{\partial} e' + \partial e'' + e) = (a, a', a'', c).$$

This proves $\mathrm{id}^{\oplus 3} \oplus \omega(n)$ is surjective. Thus, we can construct $\mu = (\alpha, \alpha', \alpha'', \beta)(e)$ by choosing a preimage of $\lambda(e)$ through $(\mathrm{id}^{\oplus 3} \oplus \omega(n))$. In particular, we obtain a map $\beta : E \to \mathcal{Q}(n)$ such that $-(\partial + \bar{\partial})\beta(e) = (\alpha'' + \alpha')(e) = -\varphi(\partial + \bar{\partial})(e)$. According to Lemma 2.4.2, we may obtain ψ' extending φ from $\beta : E \to \mathcal{Q}(n)$. Moreover, $\omega\beta = \psi|_E$. $\square$

Definition 2.4.4. We will say that an operad is *cofibrant* if is the colimit of a sequence of principal extensions starting from the initial operad I.

The following lifting lemma justifies the above definition.

Proposition 2.4.5. *Let $\mathcal{O}$ be a cofibrant operad. For every diagram of operads*

$$\begin{tikzcd} & \mathcal{P} \arrow[d, two heads, "\omega", "\wr"'] \\ \mathcal{O} \arrow[r, "\varphi"] \arrow[ur, dashed, "\varphi'"] & \mathcal{Q} \end{tikzcd}$$

in which ω is a surjective weak equivalence, there exists a morphism of operads φ' making the diagram commute.

Proof. We construct φ' by induction using Lemma 2.4.3. $\square$

Remark 2.4.6. Let $\mathcal{P}$ be an operad and K a commutative bb-algebra. Then $\mathcal{P} \otimes K = \{\mathcal{P}(n) \otimes K\}_{n \geq 0}$ has a natural operad structure given by $(p \otimes a) \circ_i (q \otimes b) = (-1)^{|a||q|}(p \circ_i q) \otimes (ab)$.

Consider the cbba $R = \mathbf{k}[t, \partial t, \bar{\partial} t, \partial\bar{\partial} t] = \Lambda(t, \partial t, \bar{\partial} t, \partial\bar{\partial} t)$, where $|t| = (0,0)$. We have the unit $\iota \colon \mathbf{k} \to R$ and evaluations δ^0 and δ^1 at $t = 0$ and $t = 1$, respectively. These morphisms satisfy $\delta^0 \circ \iota = \delta^1 \circ \iota = 1$. By [Ste25a, Lemma 2.13], ι is a weak equivalence of cbba's.

Definition 2.4.7. The *functorial path* in the category of operads is defined as the functor $- \otimes R : \mathrm{Op(BiCo)} \to \mathrm{Op(BiCo)}$, given on objects by $\mathcal{P}[t, \partial t, \bar{\partial} t, \partial\bar{\partial} t] = \mathcal{P} \otimes R$ and on morphisms by $\varphi \otimes \mathrm{id}_R$ together with the natural transformations

$$\mathcal{P} \xrightarrow{\ \iota\ } \mathcal{P} \otimes R \overset{\delta^0}{\underset{\delta^1}{\rightrightarrows}} \mathcal{P}$$

such that $\delta^k \circ \iota = \mathrm{id}$, where $\delta^k = \mathrm{id} \otimes \delta^k \colon \mathcal{P} \otimes R \to \mathcal{P} \otimes \mathbf{k} = \mathcal{P}$ and $\iota = \mathrm{id} \otimes \iota \colon \mathcal{P} = \mathcal{P} \otimes \mathbf{k} \to \mathcal{P} \otimes R$.

Since $\iota \colon \mathbf{k} \to R$ is a pluripotential weak equivalence, we have $\iota(n) \colon \mathcal{P}(n) = \mathcal{P}(n) \otimes \mathbf{k} \to \mathcal{P}(n) \otimes R$ is also a pluripotential weak equivalence of bicomplexes and hence, $\iota \colon \mathcal{P} \to \mathcal{P} \otimes R$ is a weak equivalence of operads. Note also that the path preserves weak equivalences. Also, the map $(\delta^0, \delta^1) \colon \mathcal{P} \otimes R \to \mathcal{P}$ is surjective. The maps δ^0 and δ^1 are surjective weak equivalences.

Definition 2.4.8. Let $\varphi, \psi \colon \mathcal{P} \to \mathcal{Q}$ be two morphisms of operads. A *homotopy from φ to ψ* is given by a morphism of operads $H \colon \mathcal{P} \to \mathcal{Q} \otimes R$ such that $\delta^0 \circ H = \varphi$ and $\delta^1 \circ H = \psi$. We use the notation $H \colon \varphi \simeq \psi$.

We say that φ and ψ are *homotopic* if there exists a homotopy $H \colon \varphi \simeq \psi$.

Lemma 2.4.9. *Let $\varphi, \psi \colon \mathcal{P} \to \mathcal{Q}$ be two homotopic morphisms of operads, then the maps $\varphi(n), \psi(n) \colon \mathcal{P}(n) \to \mathcal{Q}(n)$ are homotopic as maps of bicomplexes.*

Proof. We have the following decomposition $(\mathcal{Q} \otimes R)(n) = \mathcal{Q}(n) \oplus (\mathcal{Q} \otimes R^+)(n)$. Here, R^+ denotes the augmentation ideal of R. Note that $\delta^0 = \delta^1$ in the first summand. Which implies that $\varphi - \psi$ factors through $(R^+ \otimes \mathcal{Q})(n)$ which is a direct sum of squares in every arity, therefore $(\varphi - \psi)(n)$ is null-homotopic (see [Ste25a, Lemma 1.5]) and hence $\varphi(n) \simeq \psi(n)$. $\square$

The homotopy relation is symmetric and reflexive. We next prove that it is transitive if the source operad is cofibrant.

Proposition 2.4.10. *The homotopy relation between morphisms of operads is an equivalence relation for those morphisms whose source is a cofibrant operad.*

Proof. We prove transitivity, adapting the proof of [CR19] for operadic algebras. Let $\mathcal{O}$ be a cofibrant operad and consider morphisms $f, f', f'' : \mathcal{O} \to \mathcal{P}$ together with homotopies $H : f \simeq f'$ and $H' : f' \simeq f''$. Consider the pullback diagram in the category of operads

$$\begin{tikzcd}
\mathcal{P}[t, \partial t, \bar\partial t, \partial\bar\partial t, s, \partial s, \bar\partial s, \partial\bar\partial s] \arrow[drr, dashed, "\pi"] \arrow[drrr, bend left, "\delta^1_s"] \arrow[ddr, bend right, "\delta^0_t"'] & & & \\
& & \mathcal{Q} \arrow[r] \arrow[d] & \mathcal{P}[t, \partial t, \bar\partial t, \partial\bar\partial t] \arrow[d, "\delta^0_t"] \\
& \mathcal{P}[s, \partial s, \bar\partial s, \partial\bar\partial s] \arrow[rr, "\delta^1_s"'] & & \mathcal{P}
\end{tikzcd}$$

Since δ^i_* are surjective weak equivalences, one may verify that π is a surjective weak equivalence.

Consider the diagram

$$\begin{tikzcd}
& \mathcal{P}[t, \partial t, \bar\partial t, \partial\bar\partial t, s, \partial s, \bar\partial s, \partial\bar\partial s] \arrow[d, two heads, "\pi", "\wr"'] \\
\mathcal{O} \arrow[ur, dashed, "g"] \arrow[r, "{(H,H')}"'] & \mathcal{Q}\,.
\end{tikzcd}$$

By Proposition 2.4.5, there exists a map g such that $\pi g = (H, H')$. Consider the map $\nabla : \mathcal{P}[t, \partial t, \bar\partial t, \partial\bar\partial t, s, \partial s, \bar\partial s, \partial\bar\partial s] \to \mathcal{P}[t, \partial t, \bar\partial t, \partial\bar\partial t]$ given by $t, s \mapsto t$. Then ∇g gives the desired homotopy $\nabla g : f \simeq f''$. ☐

We will also use the following factorization lemma, which is a matter of verification. Let $\varphi \colon \mathcal{P} \to \mathcal{Q}$ be a morphism of operads. Define the mapping path of φ by the following pull-back

$$\begin{tikzcd}
\mathcal{M}(\varphi) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Q}[t, \partial t, \bar\partial t, \partial\bar\partial t] \arrow[d, "\delta^0_{\mathcal{Q}}"] \\
\mathcal{P} \arrow[r, "\varphi"] & \mathcal{Q}\,.
\end{tikzcd}$$

We have

Lemma 2.4.11 (c.f. [Bro73])**.** *Given a map of operads $\varphi : \mathcal{P} \to \mathcal{Q}$, there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{P} & \mathcal{M}(\varphi) \arrow[l, "\pi_1"'] \arrow[r, "\delta^1_{\mathcal{Q}}\pi_2"] & \mathcal{Q} \\
& \mathcal{P} \arrow[ul, equal] \arrow[u, "\iota_\varphi"] \arrow[ur, "\varphi"'] &
\end{tikzcd},$$

where $\iota_\varphi := (\mathrm{id}_{\mathcal{P}}, \iota_{\mathcal{Q}}\varphi)$. In addition, π_1 is a surjective weak equivalence and $\delta^1_{\mathcal{Q}}\pi_2$ is surjective. In particular, ι_φ is a weak equivalence. If φ is a weak equivalence, then $\delta^1_{\mathcal{Q}}\pi_2$ is a weak equivalence.

Given operads $\mathcal{O}$, $\mathcal{P}$, with $\mathcal{O}$ cofibrant, denote by $[\mathcal{O},\mathcal{P}]$ the set of homotopy classes of morphisms of operads $\mathcal{O}\to\mathcal{P}$. We have:

Proposition 2.4.12. *Let $\mathcal{O}$ be a cofibrant operad. Any weak equivalence $\omega\colon\mathcal{P}\to\mathcal{Q}$ of operads induces a bijection $\omega_*\colon[\mathcal{O},\mathcal{P}]\to[\mathcal{O},\mathcal{Q}]$.*

Proof. We follow closely the proof given in [CR19] for operadic algebras. We first will prove ω_* is surjective. By Lemma 2.4.11, we have the following diagram

$$\begin{tikzcd}
 & \mathcal{P} \arrow[d, shift right, "{(\mathrm{id},\iota\omega)}"'] \arrow[dd, bend left=60, "\omega"] \\
 & \mathcal{M}(\omega) \arrow[u, shift right, "\pi_1"'] \arrow[d, two heads, "\delta^1\pi_2", "\wr"'] \\
\mathcal{O} \arrow[ur, dashed, "g'"] \arrow[r, "f"] & \mathcal{Q}
\end{tikzcd}$$

where $\delta^1\pi_2(\mathrm{id},\iota\omega)=\omega$ and $\delta^1\pi_2$ is a surjective weak equivalence. By Proposition 2.4.5, there exists g' such that $\delta_1\pi_2 g'=f$. Define $g:=\pi_1 g'$ then, $f=\delta_1\pi_2 g'$ and $\omega g=\omega\pi_1 g'=\delta^0\pi_2 g'$, which implies $\pi_2 g'\colon f\simeq\omega g$. Therefore, for every $f\colon\mathcal{O}\to\mathcal{Q}$ there exists $g\colon\mathcal{O}\to\mathcal{P}$ such that $[\omega g]=[f]$ and ω_* is surjective.

Let us prove injectivity. Let $f_0,f_1\colon\mathcal{O}\to\mathcal{P}$ be such that $h\colon\omega f_0\simeq\omega f_1$. Consider the pull-back diagram:

$$\begin{tikzcd}
\mathcal{P}[t,\partial t,\bar\partial t,\partial\bar\partial t] \arrow[dr, dashed, "\overline{\omega}"] \arrow[drr, bend left, "{\omega[t,\partial t,\bar\partial t,\partial\bar\partial t]}"] \arrow[ddr, bend right, "{(\delta^0,\delta^1)}"'] & & \\
 & \mathcal{M}(\omega,\omega) \arrow[r] \arrow[d] & \mathcal{Q}[t,\partial t,\bar\partial t,\partial\bar\partial t] \arrow[d, "{(\delta^0,\delta^1)}"] \\
 & \mathcal{P}\times\mathcal{P} \arrow[r, "\omega\times\omega"] & \mathcal{Q}\times\mathcal{Q}
\end{tikzcd}$$

One can verify that $\overline{\omega}$ is a weak equivalence, see [Cir15, Lemma 2.18]. Let $H=(f_0,f_1,h)$ and consider the following diagram:

$$\begin{tikzcd}
 & \mathcal{P}[t,\partial t,\bar\partial t,\partial\bar\partial t] \arrow[d, "\overline{\omega}"] \\
\mathcal{O} \arrow[ur, dashed, "G"] \arrow[r, "H"] & \mathcal{M}(\omega,\omega).
\end{tikzcd}$$

Since $\overline{\omega}_*$ is surjective, there exists a dotted arrow G such that $\overline{\omega}G\simeq H$. It follows that $f_0\simeq\delta^0 G\simeq\delta^1 G\simeq f_1$. Then $f_0\simeq f_1$ by Proposition 2.4.10. $\square$

We next define minimal operads as cofibrant operads satisfying an additional decomposability condition on their differentials, and prove their existence and uniqueness. Recall that the free operad is weight graded and that $\mathbb{T}(M)^{(r)}$ denotes the $\mathbb{S}$-module of operations of weight r, see Remark 2.2.13.

Definition 2.4.13. A *minimal* operad is a cofibrant operad $\mathcal{M}$ such that the differentials are minimal, in the sense that $\partial\bar\partial(\mathcal{M})\subseteq\mathcal{M}^{(\geq 2)}$.

The following lemma is a key result for building minimal models using principal extensions, we will use the cone of a map of bicomplexes defined in Section 1.1.

Lemma 2.4.14. *Let $\alpha : \mathcal{M} \to \mathcal{P}$ be a morphism of operads where*

- *$\mathcal{M}$ is a free operad,*
- *$\alpha : \mathcal{M} \to \mathcal{P}$ is a weak equivalence in arities $< n$.*

Then, there is an arity n principal extension $\mathcal{M}' = \mathcal{M} \sqcup_{d+\bar{d}} \mathbb{T}(E)$ and a map of operads $\alpha' : \mathcal{M}' \to \mathcal{P}$ such that it is a weak equivalence for arities $< n$ and

1. *the bicomplex $C(\alpha'(n))$ satisfies the $\partial\bar{\partial}$-property,*
2. *if $C(\alpha(n))$ already satisfies the $\partial\bar{\partial}$-property, then $\alpha'(n) : \mathcal{M}'(n) \to \mathcal{P}(n)$ is a weak equivalence for arities $\leq n$.*

Proof. Consider the bigraded $\mathbb{S}$-mod $E := H_{\mathcal{BC}}(C(\alpha(n)))$, where

$$C(\alpha(n)) = \mathcal{M}(n)[-1,-1] \oplus \mathcal{M}(n)[-1,0] \oplus \mathcal{M}(n)[0,-1] \oplus \mathcal{P}(n)$$

denotes the cone of the morphism of bicomplexes $\alpha(n)$. Consider a section $s := (d, -d_1, -d_2, f)$ of the projection

$$Z(C(\alpha(n))) \to H_{\mathcal{BC}}(C(\alpha(n))).$$

Note that $s(e) \in Z(C(\alpha(n)))$ implies

$$(\partial d, \partial d_1, \partial d_2 - d, \partial f - \alpha d_1) = 0 \Rightarrow \begin{cases} \partial d = 0 \\ \partial d_1 = 0 \\ \partial d_2 = d \\ \partial f = \alpha d_1 \end{cases}$$

and

$$(\bar{\partial} d, \bar{\partial} d_1 + d, \bar{\partial} d_2, \bar{\partial} f - \alpha d_2) = 0 \Rightarrow \begin{cases} \bar{\partial} d = 0 \\ \bar{\partial} d_1 = -d \\ \bar{\partial} d_2 = 0 \\ \bar{\partial} f = \alpha d_2 \end{cases}$$

This shows that $(d_1 + d_2)$ defines a differential $\partial_{M'} + \bar{\partial}_{M'}$ on $\mathcal{M}' = \mathcal{M} \sqcup_{d_1+d_2} \mathbb{T}(E)$ and f extends α by Lemma 2.4.2, i.e., there exists $\alpha' : \mathcal{M}' \to \mathcal{P}$ such that $\alpha'_{|\mathcal{M}} = \alpha$ and $\alpha'|E = f$.

We now prove (1), that is we prove that $C(\alpha'(n))$ satisfies the $\partial\bar{\partial}$-property, which is equivalent to

$$\mathrm{id}_* : H_{\mathcal{BC}}(C(\alpha'(n))) \to H_{\mathcal{A}}(C(\alpha'(n))) \tag{2.3}$$

being an isomorphism, see Section 1.1.

By Remark 1.1.4, it is enough to see that $\widetilde{H}_{\mathcal{BC}}(C(\alpha'(n))) = 0$, where we recall that

$$\widetilde{H}_{\mathcal{BC}} := \frac{\mathrm{Im}\partial \cap \ker \bar{\partial} + \mathrm{Im}\bar{\partial} \cap \ker \partial}{\mathrm{Im}\partial\bar{\partial}}.$$

We have
$$C(\alpha'(n)) \cong \mathsf{L} \otimes \mathcal{M}'(n) \oplus \mathcal{P}(n) = \mathsf{L} \otimes (\mathcal{M}(n) \oplus E) \oplus \mathcal{P}(n),$$
then, every element in $C(\alpha'(n))$ can be expressed as $(a+e, a'+e', a''+e'', c)$. Let $x = [(a+e, a'+e', a''+e'', c)] \in H_{\mathcal{BC}}(C(\alpha'(n)))$, the cycle condition gives
$$\begin{cases} \partial(a+e) = \bar\partial(a+e) = 0, \\ -\partial(a'+e') = 0, \\ \bar\partial(a'+e') = a+e, \\ -\partial(a''+e'') = a+e, \\ \bar\partial(a''+e'') = 0, \\ \partial c = -\alpha'(a'+e'), \\ \bar\partial c = -\alpha'(a''+e''). \end{cases}$$
Which implies $e = 0$, since $\partial(a'+e') \in \mathcal{M}(n)$. Graphically, this means
$$\begin{tikzcd}
\bullet^{a''+e''} \arrow[r, dash] \arrow[rr, bend left, "\alpha'"] & \bullet^{a} \arrow[d, dash] & \bullet^{\bar\partial c} \arrow[d, dash] & \\
 & \bullet^{a'+e'} \arrow[rr, bend right, "\alpha'"'] & \bullet^{c} \arrow[r, dash] & \bullet^{\partial c}.
\end{tikzcd}$$
We want to show that if a representative of a Bott-Chern class of $C(\alpha'(n))$ is in $\mathrm{Im}\partial + \mathrm{Im}\partial$ then it is in $\mathrm{Im}\partial\bar\partial$ and thus, its class is trivial. Suppose there is an element $(z, z', z'', c') \in C(\alpha'(n))$ such that
$$\partial(z, z', z'', c') = (a, a'+e', a''+e'', c),$$
then
$$\begin{cases} \partial z = a, \\ -\partial z' = a'+e', \\ -\partial z'' - z = a''+e'', \\ \partial c' + \alpha'(z') = c, \end{cases}$$
which implies $e' = 0$ since $\partial z' \in \mathcal{M}(n)$ and $\partial\bar\partial z' = \bar\partial a' = a$. Now, since $e'' \in E = H_{\mathcal{BC}}(C(\alpha(n)))$, we have, by definition, that
$$e'' = [s(e'')] = [(-\partial\bar\partial a'', \partial a'' + a, \bar\partial a'', -\bar\partial c - \alpha(a''))].$$
Note that
$$(-\bar\partial z' - a'', 0, 0, c') \in C(\alpha(n))$$
and
$$\begin{aligned} \partial\bar\partial(-a'' - \bar\partial z', 0, 0, c') &= (-\partial\bar\partial a'', \partial a'' + \partial\bar\partial z', \bar\partial a'', \partial\bar\partial c' - \alpha(a'') - \alpha(\bar\partial z')) = \\ &= (-\partial\bar\partial a'', \partial a'' + a, \bar\partial a'', -\bar\partial c - \alpha(a'')) = s(e''). \end{aligned}$$
This implies $e'' = 0$. Therefore, $x = [(a, a', a'', c)] \in H_{\mathcal{BC}}(C(\alpha(n))) = E$. Consequently,

$\partial\bar{\partial}(x,0,0,0) = (a,a',a'',c)$ in $C(\alpha'(n))$, indicating that x is trivial. A similar argument shows that if there is an element $(z,z',z'',c') \in C(\alpha'(n))$ such that $\bar{\partial}(z,z',z'',c') = (a,a'+e',a''+e'',c)$, then we also have that $x = [(a,a'+e',a''+e'',c)]$ is trivial. Thus, we can conclude $\widetilde{H}_{\mathcal{BC}}(C(\alpha'(n))) = 0$.

We now prove (2). Suppose $C(\alpha(n))$ satisfies the $\partial\bar{\partial}$-property. We want to show $\alpha'(n)$ is a pluripotential weak equivalence for $m = 0,\dots,n$. Since $\alpha'(m) = \alpha(m)$ for $m = 0,\dots,n-1$, it is a pluripotential weak equivalence by induction hypothesis. Let us prove that

$$\alpha'(n) = (\alpha \oplus f) : \mathcal{M}'(n) = \mathcal{M}(n) \oplus E \to \mathcal{P}(n)$$

is a pluripotential weak equivalence.

Consider the map

$$j : \mathcal{M}(n) \hookrightarrow \mathcal{M}'(n) = \mathcal{M}(n) \oplus E,$$

and the morphism

$$(\mathrm{id} \oplus \alpha') : C(j) = \mathsf{L} \otimes \mathcal{M}(n) \oplus (\mathcal{M}(n) \oplus E) \to C(\alpha(n)) = \mathsf{L} \otimes \mathcal{M}(n) \oplus \mathcal{P}(n).$$

We will prove that $H_{\mathcal{BC}}(\mathrm{id} \oplus \alpha')$ and $H_{\mathcal{A}}(\mathrm{id} \oplus \alpha')$ are isomorphisms.

Let $[(x,x',x'',y+e)] \in H_{\mathcal{BC}}(C(j))$, then

$$\begin{cases} \bar{\partial}x' = x, \\ -\partial x'' = x, \\ j(x') = x' = -\partial(y+e), \\ j(x'') = x'' = -\bar{\partial}(y+e). \end{cases}$$

Therefore,

$$\begin{aligned} [(x,x',x'',y+e)] &= [(\partial\bar{\partial}(y+e), -\partial(y+e), -\partial(y+e), y+e)] = \\ &= [(\partial\bar{\partial}e, -\partial e, -\partial e, e)] + [(\partial\bar{\partial}y, -\partial y, -\bar{\partial}y, y)] = \\ &= [(\partial\bar{\partial}e, \partial e, -\partial e, e)] + [\partial\bar{\partial}(y,0,0,0)]. \end{aligned}$$

This implies that every element in $H_{\mathcal{BC}}(C(j))$ has a representative $[(\bar{\partial}\partial e, \partial e, -\bar{\partial}e, e)]$, where $e \in E$. If $(\mathrm{id} \oplus \alpha')(\partial\bar{\partial}e, -\partial e, -\bar{\partial}e, e) = (\partial\bar{\partial}e, -\partial e, -\bar{\partial}e, \alpha' e) = s(e) = \partial\bar{\partial}(y,y',y'',c)$, then $e = 0$. This proves $H_{\mathcal{BC}}(\mathrm{id} \oplus \alpha')$ is injective.

Let $e \in H_{\mathcal{BC}}(C(\alpha(n)))$, we can choose a representative $e = [(x,x',x'',c)]$, which implies $\partial\bar{\partial}(e) = x$, $\partial(e) = -x'$, $\bar{\partial}(e) = -x''$ and $\alpha'(e) = c$. Then

$$[(x,x',x'',e)] \mapsto [(x,x',x'',c)].$$

This proves $H_{\mathcal{BC}}(\mathrm{id} \oplus \alpha')$ is surjective.

Now, since the map $C(\alpha(n))$ satisfies the $\partial\bar{\partial}$-property, $\mathrm{id}_*\colon H_{\mathcal{A}}(C(\alpha(n))) \to H_{\mathcal{BC}}(C(\alpha(n)))$ is an isomorphism. We have the following commutative diagram:

$$\begin{tikzcd} H_{\mathcal{BC}}(C(j)) \arrow[r, "\mathrm{id}_*"] \arrow[d, "\cong"'] & H_{\mathcal{A}}(C(j)) \arrow[d] \\ H_{\mathcal{BC}}(C(\alpha(n))) \arrow[r, "\mathrm{id}_*", "\cong"'] & H_{\mathcal{A}}(C(\alpha(n))) \end{tikzcd}$$

which implies $\mathrm{id}_* : H_{\mathcal{BC}}(C(j)) \to H_{\mathcal{A}}(C(j))$ is injective and, hence, an isomorphism, see Remark 1.1.3. This gives us the isomorphism $H_{\mathcal{A}}(\mathrm{id} \oplus \alpha')$.

Consider the following commutative diagram where the rows are exact, see [Ste25a, Corollary 1.12]:

$$\begin{tikzcd} H^{p-1,q-1}_{\mathcal{A}}(\mathcal{M}(n)) \arrow[r] \arrow[d, "\cong"'] & H^{p-1,q-1}_{\mathcal{A}}(\mathcal{M}'(n)) \arrow[r] \arrow[d, "\alpha'"] & H^{p-1,q-1}_{\mathcal{A}}(C(j)) \arrow[r] \arrow[d, "\cong"', "(\mathrm{id}\oplus\alpha')"] & H^{pq}_{\mathcal{BC}}(\mathcal{M}(n)) \arrow[r] \arrow[d, "\cong"'] & H^{pq}_{\mathcal{BC}}(\mathcal{M}'(n)) \arrow[r] \arrow[d, "\alpha'"] & H^{pq}_{\mathcal{BC}}(C(j)) \arrow[d, "\cong"', "(\mathrm{id}\oplus\alpha')"] \\ H^{p-1,q-1}_{\mathcal{A}}(\mathcal{M}(n)) \arrow[r] & H^{p-1,q-1}_{\mathcal{A}}(\mathcal{P}(n)) \arrow[r] & H^{p-1,q-1}_{\mathcal{A}}(C(\alpha(n))) \arrow[r] & H^{pq}_{\mathcal{BC}}(\mathcal{M}(n)) \arrow[r] & H^{pq}_{\mathcal{BC}}(\mathcal{P}(n)) \arrow[r] & H^{pq}_{\mathcal{BC}}(C(\alpha(n))) \end{tikzcd}$$

by the Five Lemma, $H_{\mathcal{A}}(\alpha'(n))$ and $H_{\mathcal{BC}}(\alpha'(n))$ are isomophisms, which shows that $\alpha'(n)$ is a pluripotential weak equivalence. □

Theorem 2.4.15 (Existence of minimal models)**.** *Let $\mathcal{P}$ be an operad such that $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \mathbf{k}$. There exists a minimal operad $\mathcal{M}$ and a weak equivalence $\mathcal{M} \xrightarrow{\sim} \mathcal{P}$.*

Proof. We will define, inductively on arity $m \geq 2$, a morphism of operads

$$\alpha_m : \mathcal{M}_m \to \mathcal{P}$$

in such a way that:

(i) $\mathcal{M}_m$ is a minimal operad generated in arities $< m$,

(ii) $\alpha_m \colon \mathcal{M}_m \to \mathcal{P}$ is a pluripotential weak equivalence in arities $< m$.

For our base case, we begin by adding all zig-zags in arity 2: let E be the bb-$\mathbb{S}$-module $E = H_{\mathcal{ABC}}(\mathcal{P}(2))$ and $s_2 : H_{\mathcal{ABC}}(\mathcal{P}(2)) \to \mathcal{P}(2)$ a section of a projection $\mathcal{P}(2) \cong H_{\mathcal{ABC}}(\mathcal{P}(2)) \oplus \mathcal{P}(2)_{sq} \twoheadrightarrow H_{\mathcal{ABC}}(\mathcal{P}(2))$. We set

$$\mathcal{M}_2 = \mathbb{T}(E), \quad \partial_{2|E} = \partial_{H_{\mathcal{ABC}}} \quad \bar{\partial}_{2|E} = \bar{\partial}_{H_{\mathcal{ABC}}} \quad \alpha_2 : \mathcal{M}_2 \to \mathcal{P},\ \alpha_{2|E} = s_2.$$

Note that $\partial_{2|E}\bar{\partial}_{2|E} = 0$. The map α_2 is a morphism of operads and is a pluripotential weak equivalence in arities ≤ 2.

Assume we have defined $\alpha_m \colon \mathcal{M}_m \to \mathcal{P}$ for all $m < n$. Let us now build $\alpha_n : \mathcal{M}_n \to \mathcal{P}$. We apply Lemma 2.4.14 to construct an arity n principal extension $\mathcal{M}' = \mathcal{M}_{n-1} \sqcup_{d'+\overline{d'}} \mathbb{T}(E')$ and obtain a map $\alpha' : \mathcal{M}' \to \mathcal{P}$ such that $C(\alpha'(n))$ satisfies the $\partial\bar{\partial}$-property.

Then, $\mathcal{M}'$ is minimal since it is cofibrant and $d_{\mathcal{M}'}(\mathcal{M}') = (\partial_{\mathcal{M}'} + \bar{\partial}_{\mathcal{M}'})(\mathcal{M}') \subseteq \mathcal{M}'^{(\geq 2)}$. Indeed,

- $d_{\mathcal{M}'|\mathcal{M}_{n-1}} = d_{\mathcal{M}_{n-1}}$, but $\partial_{\mathcal{M}_{n-1}}, \bar{\partial}_{\mathcal{M}_{n-1}}$ are minimal by induction hypothesis,
- $d_{\mathcal{M}'|E'} : E' \xrightarrow{d'+\overline{d}'} \mathcal{M}_{n-1}(n) \subseteq \mathcal{M}'^{(\geq 2)}$.

We now apply again Lemma 2.4.14 to the map $\alpha'\colon \mathcal{M}' \to \mathcal{P}$, which satisfies that $C(\alpha'(n))$ has the $\partial\bar{\partial}$-property. This produces a map $\alpha_n\colon \mathcal{M}_n = \mathcal{M}' \sqcup_{d_n+\bar{d}_n} \mathbb{T}(E) \to \mathcal{P}$ which is a pluripotential weak equivalence in arities $\leq n$.

Again, $\mathcal{M}_n$ is minimal since it is cofibrant and $\partial_{\mathcal{M}_n}\bar{\partial}_{\mathcal{M}_n}(\mathcal{M}_n) \subseteq \mathcal{M}_n^{(\geq 2)}$. Indeed,

- $(\partial_{\mathcal{M}_n}\bar{\partial}_{\mathcal{M}_n})_{|\mathcal{M}'} = \partial_{\mathcal{M}'}\bar{\partial}_{\mathcal{M}'}$, but $\partial_{\mathcal{M}'}, \bar{\partial}_{\mathcal{M}'}$ are already minimal and

- $(\partial_{\mathcal{M}_n}\bar{\partial}_{\mathcal{M}_n})_{|E} : E \xrightarrow{\bar{d}_n} \mathcal{M}_{n-1}(n) \oplus E' \xrightarrow{\partial \oplus d'} \mathcal{M}_{n-1}(n) \subseteq \mathcal{M}_n^{(\geq 2)}$.

This completes the proof. □

Proposition 2.4.16. *A map $\phi\colon \mathcal{M} \to \mathcal{N}$ of minimal operads $\mathcal{M} = (\mathbb{T}(M), \partial, \bar{\partial})$ and $\mathcal{N} = (\mathbb{T}(N), \partial, \bar{\partial})$ is an isomorphism if and only if it is a weak equivalence.*

Proof. Let us first introduce some notation. Let $M = \{M(n)\}_{n\geq 1}$ be a bb-$\mathbb{S}$-module and let $\mathcal{M} = \mathbb{T}(M)$. We will denote $M_{<k}$ the $\mathbb{S}$-module defined by

$$M_{<k}(n) := \begin{cases} M(n), & \text{if } k < n, \\ 0, & \text{otherwise,} \end{cases}$$

and $M_{\leq k}$ the $\mathbb{S}$-module defined by

$$M_{\leq k}(n) := \begin{cases} M(n), & \text{if } k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, $\mathcal{M}_{\leq k} := \mathbb{T}(M_{\leq k})$.

Now, one implication is clear since an isomorphism is already a weak equivalence. Let us prove the other implication: We will prove, by induction, that the restriction $\phi_n : \mathcal{M}_{\leq n} \to \mathcal{N}_{\leq n}$, is an isomorphism for each n. This will imply that $\phi : \mathbb{T}(M) \to \mathbb{T}(N)$ is an isomorphism too.

Since $\mathcal{M} = (\mathbb{T}(M), \partial, \bar{\partial})$ and $\mathcal{N}(\mathbb{T}(N), \partial, \bar{\partial})$ are minimal, we have that $\partial\bar{\partial}(\mathcal{M}(2)) = 0$ and $\partial\bar{\partial}(\mathcal{N}(2)) = 0$. Since a pluripotential weak equivalence between minimal bicomplexes is an isomorphism, $\phi_2 : \mathcal{M}(2) \to \mathcal{N}(2)$ is an isomorphism. Now, suppose we have already proved that ϕ_n is an isomorphism. We have two short exact sequences

$$0 \to \mathcal{M}_{\leq n}(n+1) \to \mathcal{M}(n+1) \to M(n+1) \to 0,$$

and

$$0 \to \mathcal{N}_{\leq n}(n+1) \to \mathcal{N}(n+1) \to N(n+1) \to 0.$$

As a result, by Lemma 1.1.5, we obtain the following two commutative diagrams, where the rows are exact:

$$\begin{tikzcd} H_{\mathcal{A}}(\mathcal{M}_{\leq n}(n+1)) \arrow[r] \arrow[d, "\cong"'] & H_{\mathcal{A}}(\mathcal{M}(n+1)) \arrow[r] \arrow[d, "\cong"'] & H_{\mathcal{A}}(M(n+1)) \arrow[r] \arrow[d] & H_{\mathcal{BC}}(\mathcal{M}_{\leq n}(n+1)) \arrow[r] \arrow[d, "\cong"'] & H_{\mathcal{BC}}(\mathcal{M}(n+1)) \arrow[d, "\cong"'] \\ H_{\mathcal{A}}(\mathcal{N}_{\leq n}(n+1)) \arrow[r] & H_{\mathcal{A}}(\mathcal{N}(n+1)) \arrow[r] & H_{\mathcal{A}}(N(n+1)) \arrow[r] & H_{\mathcal{BC}}(\mathcal{N}_{\leq n}(n+1)) \arrow[r] & H_{\mathcal{BC}}(\mathcal{N}(n+1)) \end{tikzcd}$$

and

$$\begin{tikzcd}
H_{\mathcal{BC}}(\mathcal{M}_{\leq n}(n+1)) \arrow[r] \arrow[d, "\cong"'] & H_{\mathcal{BC}}(\mathcal{M}(n+1)) \arrow[r] \arrow[d, "\cong"'] & H_{\mathcal{BC}}(M(n+1)) \arrow[r] \arrow[d] & H_{\mathcal{B}_{p,q}}(\mathcal{M}_{\leq n}(n+1)) \arrow[r] \arrow[d, "\cong"'] & H_{\mathcal{B}_{p,q}}(\mathcal{M}(n+1)) \arrow[d, "\cong"'] \\
H_{\mathcal{BC}}(\mathcal{N}_{\leq n}(n+1)) \arrow[r] & H_{\mathcal{BC}}(\mathcal{N}(n+1)) \arrow[r] & H_{\mathcal{BC}}(N(n+1)) \arrow[r] & H_{\mathcal{B}_{p,q}}(\mathcal{N}_{\leq n}(n+1)) \arrow[r] & H_{\mathcal{B}_{p,q}}(\mathcal{N}(n+1))
\end{tikzcd}$$

by the Five Lemma, it follows that $\phi(n+1)\colon M(n+1) \to N(n+1)$ is a pluripotenital weak equivalence. The minimality of $\mathcal{M}$ implies that the differentials ∂ and $\bar{\partial}$ on

$$M(n+1) \cong \mathcal{M}(n+1)/\mathcal{M}_{\leq n}(n+1)$$

verify $\partial\bar{\partial} = 0$, and the same holds true for $\mathcal{N}$. Again, this implies $\phi(n+1)\colon M(n+1) \to N(n+1)$ is an isomorphism. We have proved that ϕ_{n+1} induces an isomorphism on the generators, hence, the map

$$\phi_{n+1} : \mathbb{T}(M(\leq n+1)) \to \mathbb{T}(N(\leq n+1))$$

is also an isomorphism. □

Definition 2.4.17. A *minimal model* of an operad $\mathcal{P}$ in bicomplexes is given by a minimal operad $\mathcal{M}$ together with a weak equivalence $\mathcal{M} \xrightarrow{\sim} \mathcal{P}$.

Theorem 2.4.18 (Uniqueness of minimal models)**.** *Minimal models are unique up to isomorphism.*

Proof. Let $\alpha : \mathcal{M} \xrightarrow{\sim} \mathcal{P}$ and $\alpha' : \mathcal{N} \xrightarrow{\sim} \mathcal{P}$ two minimal models of an operad $\mathcal{P}$. By Proposition 2.4.12, there is a map $\phi : \mathcal{M} \to \mathcal{N}$ unique up to homotopy such that the following diagram commutes up to homotopy

$$\begin{tikzcd}
 & \mathcal{N} \arrow[d, "\wr"] \\
\mathcal{M} \arrow[ur, dashed, "\phi"] \arrow[r, "\sim"] & \mathcal{P}.
\end{tikzcd}$$

Using Lemma 2.4.9, we obtain $\alpha'(n) \circ \phi(n) \simeq \alpha(n)$ for every n and, by Lemma 1.1.11, ϕ is a weak equivalence. By Proposition 2.4.16, $\phi : \mathcal{M} \to \mathcal{N}$ is an isomorphism. □

2.5 Twisting morphisms and twisted composite products

We introduce operadic twisting morphisms for operads in bicomplexes. As in the classical setting, they play a central role in the bar-cobar constructions. We then use twisting morphisms to define twisted composite products, which lead to the notion of Koszul operads. Let $\mathcal{P}$ and $\mathcal{C}$ be an operad and a cooperad in bicomplexes.

Consider the bb-$\mathbb{S}$-module $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{P})$ given by

$$\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{P})(n)^{p,q} := \underline{\mathrm{Hom}}(\mathcal{C}(n), \mathcal{P}(n))^{p,q}$$

with right action of the symmetric group given by:

$$f^{\sigma}(x) := f(x^{\sigma^{-1}})^{\sigma}.$$

As in the dg setting, the pre-Lie product on $\underline{\mathrm{Hom}}(\mathcal{C},\mathcal{P})$ is defined as

$$f \star g := \mathcal{C} \xrightarrow{\Delta_{(1)}} \mathcal{C} \circ_{(1)} \mathcal{C} \xrightarrow{f \circ_{(1)} g} \mathcal{P} \circ_{(1)} \mathcal{P} \xrightarrow{\gamma_{(1)}} \mathcal{P}. \tag{2.4}$$

The space of invariant elements of $\underline{\mathrm{Hom}}(\mathcal{C},\mathcal{P})$ under the symmetric action

$$\underline{\mathrm{Hom}}_{\mathbb{S}}(\mathcal{C},\mathcal{P})$$

is stable under the pre-Lie product, and we obtain

Proposition 2.5.1. *The product space of* $\mathbb{S}$*-equivariant maps*

$$\left(\underline{\mathrm{Hom}}_{\mathbb{S}}(\mathcal{C},\mathcal{P}), \star, \partial, \bar{\partial}\right)$$

is a bidifferential bigraded pre-Lie algebra.

Proof. It suffices to prove that the two components ∂ and $\bar{\partial}$ are derivations with respect to $\star$. Therefore, the proof reduces to the classical case (see [LV12, Proposition 6.4.5]). □

Recall that a twisting morphism in the dg case is an equivariant map $\varphi : \mathcal{C} \to \mathcal{P}$ of degree -1 satisfying

$$[d,\varphi] + \varphi \star \varphi = 0.$$

With this in mind, we introduce the following bigraded version of a twisting morphism.

Definition 2.5.2. A *twisting morphism* $\varphi = (\varphi^{10}, \varphi^{01}) : \mathcal{C} \to \mathcal{P}$ consists of two maps $\varphi^{10}, \varphi^{01} \in \underline{\mathrm{Hom}}_{\mathbb{S}}(\mathcal{C},\mathcal{P})$ of bidegrees $(1,0)$ and $(0,1)$, respectively, satisfying:

$$\begin{cases} [\partial, \varphi^{10}] + \varphi^{10} \star \varphi^{10} = 0, \\ [\bar{\partial}, \varphi^{10}] + \varphi^{10} \star \varphi^{01} + [\partial, \varphi^{01}] + \varphi^{01} \star \varphi^{10} = 0, \\ [\bar{\partial}, \varphi^{01}] + \varphi^{01} \star \varphi^{01} = 0. \end{cases}$$

Remark 2.5.3. Let $\varphi : \mathcal{C} \to \mathcal{P}$ be an equivariant map such that it splits into two components of bidegrees $(1,0)$ and $(0,1)$, $\varphi = \varphi^{10} + \varphi^{01}$. Then, φ is a twisting morphism in the dg sense with respect to the total differential $d = \partial + \bar{\partial}$ if, and only if, $(\varphi^{10}, \varphi^{01})$ is a twisting morphism in the bigraded sense.

A *left module* over $\mathcal{P}$ is a bb-$\mathbb{S}$-module M equipped with a left action

$$\lambda : \mathcal{P} \circ M \to M$$

which commutes with the differentials.

Let N be a bb-$\mathbb{S}$-module. The *free left* $\mathcal{P}$ *module* on N is given by

$$(\mathcal{P} \circ N, \partial_{\mathcal{P}\circ N}, \bar{\partial}_{\mathcal{P}\circ N}),$$

where the action λ is given by the operadic composition. The proof of the following proposition can be found in Proposition 6.3.9 in [LV12].

Proposition 2.5.4. *Let $\mathcal{P}$ be an operad and let N be a* bb-$\mathbb{S}$*-module. There is a one-to-one correspondence between derivations on the free $\mathcal{P}$-module $\mathcal{P} \circ N$ and their restriction to the space of generators N. More precisely, the unique derivation $d_\varphi : \mathcal{P} \circ N \to \mathcal{P} \circ N$ which extends $\varphi : N \to \mathcal{P} \circ N$ is given by*

$$d_\varphi = (\gamma_{(1)} \circ \mathrm{id}_N)(\mathrm{id}_{\mathcal{P}} \circ' \varphi).$$

Any linear map $\alpha : \mathcal{C} \to \mathcal{P}$ from a cooperad $(\mathcal{C}, \Delta, \epsilon)$ to an operad $(\mathcal{P}, \gamma, u)$, defines a map

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C} \xrightarrow{\alpha \circ \mathrm{id}} \mathcal{P} \circ \mathcal{C}$$

which, by Proposition 2.5.4, induces a unique derivation on $\mathcal{P} \circ \mathcal{C}$, explicitly given by

$$d = (\gamma_{(1)} \circ \mathrm{id}_{\mathcal{C}})(\mathrm{id}_{\mathcal{P}} \circ' [(\alpha \circ \mathrm{id}_{\mathcal{C}})\Delta].$$

Proposition 2.5.5. *For any $\alpha, \beta \in \underline{\mathrm{Hom}}_{\mathbb{S}}(\mathcal{C}, \mathcal{P})$, the following relation is satisfied*

$$d_{\alpha \star \beta} = d_\alpha \circ d_\beta.$$

Proof. See [LV12, Proposition 6.4.7]. □

Let $\alpha \colon \mathcal{C} \to \mathcal{P}$ be a twisting morphism. Then, we define the *left twisted composite product* as

$$\mathcal{P} \circ_\alpha \mathcal{C} := (\mathcal{P} \circ \mathcal{C}, \partial_{\mathcal{P} \circ \mathcal{C}} + d_{\alpha^{10}}, \bar{\partial}_{\mathcal{P} \circ \mathcal{C}} + d_{\alpha^{01}}).$$

Proposition 2.5.6. *The twisted composite product defines a bicomplex, meaning the following identities hold:*

$$(\partial_{\mathcal{P} \circ \mathcal{C}} + d_{\alpha^{10}})^2 = 0, \quad (\bar{\partial}_{\mathcal{P} \circ \mathcal{C}} + d_{\alpha^{01}})^2 = 0,$$

and

$$(\partial_{\mathcal{P} \circ \mathcal{C}} + d_{\alpha^{10}})(\bar{\partial}_{\mathcal{P} \circ \mathcal{C}} + d_{\alpha^{01}}) + (\bar{\partial}_{\mathcal{P} \circ \mathcal{C}} + d_{\alpha^{01}})(\partial_{\mathcal{P} \circ \mathcal{C}} + d_{\alpha^{10}}) = 0.$$

Proof. By Remark 2.5.3, $\varphi = \varphi^{10} + \varphi^{01}$ is a dg twisting morphism. We can endow $\mathcal{P} \circ \mathcal{C}$ with a total differential of degree 1

$$d + d_\varphi = \partial_{\mathcal{P} \circ \mathcal{C}} + d_{\varphi^{10}} + \bar{\partial}_{\mathcal{P} \circ \mathcal{C}} + d_{\varphi^{01}}.$$

Thus, we have $(d + d_\varphi)^2 = d_{[d,\varphi] + \varphi \star \varphi}$, which is zero because φ is a dg twisting morphism. Since $d + d_\varphi$ splits into the two components

$$\partial_{\mathcal{P} \circ \mathcal{C}} + d_{\varphi^{10}} \quad \text{and} \quad \bar{\partial}_{\mathcal{P} \circ \mathcal{C}} + d_{\varphi^{01}}$$

of bidegree $(1, 0)$ and $(0, 1)$, by degree reasons we obtain the desired result. □

2.6 Koszul duality

In this section, we develop (pluripotential) Koszul duality for quadratic operads in $\mathrm{Vect}_{\mathbf{k}}$. To do so, we make use of the inflation functor

$$\mathrm{Inf} : \mathbb{S}\text{-}\mathrm{Ch}^{\leq 0} \to \mathbb{S}\text{-}\mathrm{BiCo}^{\neg},$$

see Section 1.5 and Section 2.1, which allows us to use many of the computations in the dg-case.

We will first define the pluripotential Koszul dual cooperad of a quadratic operad $\mathcal{P}$ in $\mathbf{k}$-vector spaces. Recall that, by definition, an *operadic quadratic data* (E, R) in $\mathrm{Vect}_{\mathbf{k}}$ consists of an $\mathbb{S}$-module E and a sub-$\mathbb{S}$-module $R \subseteq \mathbb{T}(E)^{(2)}$, E then denotes the generating space and R the set of relations. A *quadratic operad* associated to the quadratic data (E, R) is the operad $\mathcal{P}(E, R) := \mathbb{T}(E)/(R)$.

For the remainder of the section, $\mathcal{P}$ will denote a quadratic operad in $\mathrm{Vect}_{\mathbf{k}}$, we will view this operad as an operad in $\mathrm{Ch}_{\mathbf{k}}$ or in $\mathrm{BiCo}_{\mathbf{k}}$ by placing it in degree 0 or bidegree $(0, 0)$, with trivial differentials. We will denote by $\mathcal{P}^{¡} = \mathcal{C}(sE, s^2R)$ the dg Koszul dual cooperad of $\mathcal{P}$ and by Δ its cooperad structure (see Section 7.2 in [LV12]).

Following the classical case, one is led to define the pluripotential Koszul dual cooperad of $\mathcal{P}$ as $\mathcal{C}(\mathsf{L}E, \mathsf{L}_{\text{-}2}R)$. We will instead use the inflation functor, so that we can recycle the classical computations of Koszul dual cooperads.

Definition 2.6.1. Define the *pluripotential Koszul dual cooperad* of $\mathcal{P}$ as

$$\mathcal{P}^{¡}_{pp} := \mathrm{Inf}(\mathcal{P}^{¡})$$

with the cooperad structure given by

$$\Delta : \mathrm{Inf}(\mathcal{P}^{¡}) \xrightarrow{\mathrm{Inf}(\Delta)} \mathrm{Inf}(\mathcal{P}^{¡} \circ \mathcal{P}^{¡}) \xrightarrow{\psi_{\mathcal{P}^{¡},\mathcal{P}^{¡}}} \mathrm{Inf}(\mathcal{P}^{¡}) \circ \mathrm{Inf}(\mathcal{P}^{¡}),$$

where ψ denotes the colax natural transformation of Proposition 2.1.5.

Remark 2.6.2. Note that $\mathcal{P}^{¡}$ can be viewed as a sub-cooperad of $\mathcal{P}^{¡}_{pp}$, given by the elements of bidegrees $(*, 0)$ and, similarly, of bidegrees $(0, *)$.

Example 2.6.3. The pluripotential Koszul dual of As (as a non symmetric operad) is given by

$$As^{¡}_{pp}(n) = \mathsf{L}_{1-n}.$$

Denote by $\mu_n^{p,q}$ an element of arity n and bidegree (p, q). Then, the cooperad structure is given by

$$\Delta(\mu_n^{p,q}) = \sum_{i_1+\cdots+i_k=n} \sum_{\substack{p_0+\cdots+p_k=p \\ q_0+\cdots+q_k=q}} (-1)^{\theta+\varepsilon} (\mu_k^{p_0,q_0}; \mu_{i_1}^{p_1,q_1}, \ldots, \mu_{i_k}^{p_k,q_k}),$$

where $\theta = \sum_{1 \le r < s \le k} i_r(i_s + 1)$ and $\varepsilon = \sum_{j=1}^{k} (p_0 + q_0 + \cdots + p_{j-1} + q_{j-1})(p_j + q_j + i_j - 1)$. Therefore, the infinitesimal decomposition gives

$$\Delta_{(1)}(\mu_n^{p,q}) = \sum_{\substack{k+\ell=n+1 \\ 1 \le i \le k}} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} (-1)^{(i-1)(\ell+1)+\varepsilon} \mu_k^{p_1,q_1} \circ_i \mu_\ell^{p_2,q_2},$$

where $\varepsilon = (p_1 + q_1)(p_2 + q_2 + \ell + 1)$. For the symmetric version, we have

$$As^{¡}_{pp}(n) = \mathsf{L}_{1-n} \otimes \mathbf{k}[\mathbb{S}_n],$$

with the $\mathbb{S}_n$ action being trivial on L_{1-n}.

Remark 2.6.4. Note that, since $\mathcal{P}^{¡}$ is a sub-cooperad of the cofree cooperad, it is naturally weight graded. Then, the pluripotential Koszul dual $\mathcal{P}^{¡}_{pp}$ inherits a weight given by $\omega((p,q)_n \otimes \mu) = \omega(\mu)$ where μ is an element of $\mathcal{P}^{¡}$.

We will next define a twisting morphism $\kappa : \mathcal{P}^{¡}_{pp} \to \mathcal{P}$. Recall from Section 1.4 that $\mathfrak{s}_1 : \mathsf{L} \to \mathbf{k}$ is the map of bidegree $(1,0)$ projecting L to $\mathbf{k}$. Likewise, $\bar{\mathfrak{s}}_1 : \mathsf{L} \to \mathbf{k}$ is the corresponding map of bidegree $(0,1)$. We construct the following two equivariant maps:

$$\kappa^{10} \colon \mathcal{P}^{¡}_{pp} = \operatorname{Inf}(\mathcal{C}(sE, s^2R)) \twoheadrightarrow \operatorname{Inf}(sE) = \mathsf{L} \otimes E \xrightarrow{\mathfrak{s}_1 \otimes \mathrm{id}} E \hookrightarrow \mathcal{P} = \mathcal{P}(E,R)$$

and

$$\kappa^{01} \colon \mathcal{P}^{¡}_{pp} = \operatorname{Inf}(\mathcal{C}(sE, s^2R)) \twoheadrightarrow \operatorname{Inf}(sE) = \mathsf{L} \otimes E \xrightarrow{\bar{\mathfrak{s}}_1 \otimes \mathrm{id}} E \hookrightarrow \mathcal{P} = \mathcal{P}(E,R).$$

Remark 2.6.5. Recall that there is a dg twisting morphism, which by abuse of notation we will still denote by κ,

$$\kappa : \mathcal{P}^{¡} = \mathcal{C}(sE, s^2R) \twoheadrightarrow sE \to E \hookrightarrow \mathcal{P} = \mathcal{P}(E,R).$$

In fact, $\kappa^{10} = \mathfrak{s} \otimes \kappa$ and $\kappa^{01} = \bar{\mathfrak{s}} \otimes \kappa$, where $\mathfrak{s}(p,q)_n = \mathfrak{s}_1$ if $n = 1$ and zero otherwise and similarly for $\bar{\mathfrak{s}}$.

Proposition 2.6.6. *The pair $\kappa = (\kappa^{10}, \kappa^{01}) \colon \mathcal{P}^{¡}_{pp} \to \mathcal{P}$ is a twisting morphism.*

Proof. A direct computation shows

$$[\partial, \kappa^{10}] = [\bar{\partial}, \kappa^{01}] = [\partial, \kappa^{01}] + [\bar{\partial}, \kappa^{10}] = 0.$$

On the other hand, κ is only non-zero in weight 1. So the pre-Lie products $\kappa^{i,1-i} \star \kappa^{j,1-j}$, with $i,j \in \{0,1\}$ are zero except maybe in weight 2. But in weight 2 we have

$$\mathcal{P}^{¡(2)}_{pp} \to \mathcal{P}^{¡(1)}_{pp} \circ_{(1)} \mathcal{P}^{¡(1)}_{pp} \xrightarrow{\kappa^{**} \circ_{(1)} \kappa^{**}} (R) \to \mathbb{T}(E)^{(2)}/(R) = \mathcal{P}$$

which is the zero map. $\square$

Recall that a dg-operad $\mathcal{P}$ is said to be *Koszul* if and only if its associated Koszul complex is acyclic, so there is a quasi-isomorphism

$$\mathcal{P} \circ_\kappa \mathcal{P}^{¡} \xrightarrow{\sim} \mathrm{I},$$

where $\mathcal{P} \circ_\kappa \mathcal{P}^{¡}$ is the Koszul complex of a quadratic dg operad (see Section 7.4 in [LV12]). In particular, this definition depends on the chosen class of weak equivalences, which in the dg case are just quasi-isomorphisms. In the pluripotential case, we may consider the analogous definition with the corresponding weak equivalences. To this end, we construct the twisted composite product $\mathcal{P} \circ_\kappa \mathcal{P}^{¡}_{pp}$ associated to the twisting morphism $\kappa : \mathcal{P}^{¡}_{pp} \to \mathcal{P}$, see Section 2.5 for details. The following proposition shows that if $\mathcal{P}$ is a Koszul operad in the dg sense, concentrated in degree 0, then it is also Koszul in the pluripotential sense.

Proposition 2.6.7. *Let $\mathcal{P}$ be a quadratic operad in $\mathbf{k}$-vector spaces that is Koszul as a dg-operad. Then, there is a pluripotential weak equivalence*

$$\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}_{pp} \xrightarrow{\sim} \mathrm{I}.$$

Proof. By Proposition 2.1.6, the inflation functor preserves weak equivalences. Since $\mathcal{P}$ is Koszul as a dg-operad, we have a pluripotential weak equivalence

$$\mathrm{Inf}(\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}) \xrightarrow{\sim} \mathrm{I}.$$

We define a map of bicomplexes $\varphi : \mathrm{Inf}(\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}) \to \mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}_{pp}$ using the colax structure of the inflation functor given in Proposition 2.1.5

$$\varphi := \psi_{\mathcal{P},\mathcal{P}^{\text{!`}}} : \mathrm{Inf}(\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}) \to \mathcal{P} \circ_\kappa \mathrm{Inf}(\mathcal{P}^{\text{!`}}) = \mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}_{pp}$$

This is a linear map of bigraded vector spaces, and it suffices to see that it is compatible with the twisting differentials. Note that $\mathrm{Inf}(\mathcal{P}) = \mathcal{P}$ since it is concentrated in degree 0.

Since φ is constructed using the colax monoidal structure of Inf, it is clear that, when we forget the twisting differentials, we have a map of bicomplexes

$$\mathrm{Inf}(\mathcal{P} \circ \mathcal{P}^{\text{!`}}) \to \mathcal{P} \circ \mathrm{Inf}(\mathcal{P}^{\text{!`}})$$

which is a pluripotential weak equivalence by Proposition 2.1.5. Denote by d_κ and $\overline{d}_\kappa$ the twisting differentials

$$d_\kappa, \overline{d}_\kappa : \mathrm{Inf}(\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}) \to \mathrm{Inf}(\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}})$$

of bidegree $(1,0)$ and of bidegree $(0,1)$, respectively, obtained after inflation of the Koszul complex. Explicitly, these are given by

$$d_\kappa = \mathfrak{s}_n \otimes \Big(\gamma_{(1)} \circ \mathrm{id}_{\mathcal{P}^{\text{!`}}})(\mathrm{id}_{\mathcal{P}} \circ' [(\kappa \circ \mathrm{id}_{\mathcal{P}^{\text{!`}}})\Delta]\Big) \quad \text{and} \quad \overline{d}_\kappa = \bar{\mathfrak{s}}_n \otimes \Big(\gamma_{(1)} \circ \mathrm{id}_{\mathcal{P}^{\text{!`}}})(\mathrm{id}_{\mathcal{P}} \circ' [(\kappa \circ \mathrm{id}_{\mathcal{P}^{\text{!`}}})\Delta]\Big).$$

By the definition of twisted composite products (Section 2.5) and Remark 2.6.5, the derivations

$$d_{\kappa^{10}}, d_{\kappa^{01}} : \mathcal{P} \circ \mathcal{P}^{\text{!`}}_{pp} \to \mathcal{P} \circ \mathcal{P}^{\text{!`}}_{pp}$$

are given by

$$(\gamma_{(1)} \circ \mathrm{id}_{\mathcal{P}^{\text{!`}}_{pp}})(\mathrm{id}_{\mathcal{P}} \circ' [(\mathfrak{s} \otimes \kappa) \circ \mathrm{id}_{\mathcal{P}^{\text{!`}}_{pp}})\Delta] \quad \text{and} \quad (\gamma_{(1)} \circ \mathrm{id}_{\mathcal{P}^{\text{!`}}_{pp}})(\mathrm{id}_{\mathcal{P}} \circ' [(\bar{\mathfrak{s}} \otimes \kappa) \circ \mathrm{id}_{\mathcal{P}^{\text{!`}}_{pp}})\Delta].$$

Then, the map φ is a map of bicomplexes if $\varphi d_\kappa = d_{\kappa^{10}}\varphi$ and $\varphi \overline{d}_\kappa = d_{\kappa^{01}}\varphi$.

Let us check the first identity. On the one hand, we have

$$\varphi d_\kappa((p,q)_n \otimes (\mu; \nu_{i_1}, \ldots, \nu_{i_k})) =$$
$$= \sum_{j=1}^{k} \sum (-1)^{\theta + \sum_{s=1}^{j-1} d_s} (\iota \mathfrak{s}_n (p,q)_n) \otimes (\mu \circ_j \kappa \nu_{i_j}^{(1)}; \nu_{i_1}, \ldots, \nu_{i_j}^{(2)}, \ldots, \nu_{i_k}),$$

where $\nu_{i_j}^{(2)} = (\nu_{i_j,1}, \ldots, \nu_{i_j,\ell})$, θ is the sign given by the decomposition map of $\mathcal{P}^{\text{!`}}$, $d_s = |\nu_{i_s}|$ and

$$\iota = \iota_{d_1,\ldots,d_{j\,1},\ldots,d_{j\,\ell},\ldots,d_k} : \mathsf{L}_{d_1,\ldots,d_k} \to \mathsf{L}_{d_1} \otimes \cdots \otimes \mathsf{L}_{d_k},$$

defined in Section 1.4. On the other hand,

$$d_{\kappa^{10}}\varphi((p,q)_n \otimes (\mu_n; \nu_{i_1}, \ldots, \nu_{i_k})) =$$
$$= \sum_{j=1}^{k} (-1)^{\theta} ((\mathrm{id}^{j-1} \otimes (\mathfrak{s} \otimes \mathrm{id}^{\ell})\iota_{d_{j\,0},\ldots,d_{j\,\ell}} \otimes \mathrm{id}^{k-j})\iota_{d_1,\ldots,d_k}(p,q)_n)(\mu \circ_j \kappa\nu_{i_j}^{(1)}; \nu_{i_1}, \ldots, \nu_{i_j}^{(2)}, \ldots, \nu_{i_k}),$$

where $d_0 = |\nu_{i_j}^{(1)}|$ and $\mathfrak{s}(p,q)_n = \mathfrak{s}_1$ if $n = 1$ and zero otherwise. By coassociativy of the maps ι, Proposition 1.4.4, we have

$$\begin{aligned}(\mathrm{id}^{\otimes j-1} \otimes (\mathfrak{s} \otimes \mathrm{id}^{\otimes \ell})\iota_{d_{j\,0},\ldots,d_{j\,\ell}} \otimes \mathrm{id}^{\otimes k-j})\iota_{d_1,\ldots,d_k}(p,q)_n =\\ = (\mathrm{id}^{\otimes j-1} \otimes \mathfrak{s} \otimes \mathrm{id}^{\otimes k+\ell-j})\iota_{d_1,\ldots,d_{j\,0},\ldots,d_{j\,\ell},\ldots,d_k}(p,q)_n,\end{aligned}$$

which by the compatibility relations between the maps ι and $\mathfrak{s}$, Proposition 1.4.6, gives

$$(-1)^{\sum_s^{j-1} d_s}\iota_{d_1,\ldots,d_{j\,1},\ldots,d_{j\,\ell},\ldots,d_k}\mathfrak{s}_{\mathfrak{n}}(p,q)_n.$$

This proves that $\varphi d_\kappa = d_{\kappa^{10}}\varphi$. A similar computation shows $\varphi\bar{d}_\kappa = d_{\kappa^{01}}\varphi$.

This gives us a commutative diagram

$$\begin{tikzcd}
\mathcal{P} \circ_\kappa \mathrm{Inf}(\mathcal{P}^{\text{!`}}) \arrow[r] & \mathrm{I} \\
\mathrm{Inf}(\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}) \arrow[u, "\varphi"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Thus, by the two-out-of-three property, it suffices to show that φ is a pluripotential weak equivalence, which we address in the remainder of the proof.

Since $\mathcal{P}$ is concentrated in bidegree $(0,0)$, $\mathrm{Inf}(\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}})$ and $\mathcal{P} \circ_\kappa \mathrm{Inf}(\mathcal{P}^{\text{!`}})$ are bb-$\mathbb{S}$-modules in bounded bicomplexes, see Remark 1.1.9. Therefore, to show φ is a pluripotential weak equivalence is enough to show it is a quasi-isomorphism with respect to ∂ and $\bar{\partial}$. Let us focus on the ∂-case, so we forget the differential $\bar{\partial}$ and view φ as a map of complexes with respect to the ∂-component on both sides. The strategy will actually be to split the component ∂ into weights, obtaining a new bicomplex structure. Indeed, the natural weight-grading of $\mathcal{P}$ induces a weight-grading on $\mathcal{P} \circ \mathcal{P}^{\text{!`}}$ and the differential $\partial$ of $\mathrm{Inf}(\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}})$ splits into two components of weight 0 and 1 respectively $\partial = \partial_0 + \partial_1$, where $\partial_1 = d_\kappa$. So we get that $\mathrm{Inf}(\mathcal{P} \circ \mathcal{P}^{\text{!`}})$ is a bicomplex with total differential $\partial$. A similar argument shows that $\mathcal{P} \circ \mathcal{P}^{\text{!`}}_{pp}$ with its $\bar{\partial}$-differential is also a bicomplex. Moreover, φ is a map of bicomplexes with respect to the weight-grading. By Proposition 2.1.5, φ is a quasi-isomorphism with respect to the weight 0 component ∂_0 of ∂. Therefore, a standard spectral sequence argument shows it is a quasi-isomorphism with respect to the total differential. This proves that $H_\partial(\varphi)$ is an isomorphism. A parallel argument shows $H_{\bar{\partial}}(\varphi)$ is also an isomorphism. □

We will now present a concrete and computationally effective method for constructing pluripotential minimal models for operads in $\mathrm{Vect}_{\mathbf{k}}$ that are Koszul as a dg-operad, taking advantage of the dg setting.

A *coaugmented cooperad* is a cooperad $\mathcal{C}$ equipped with a morphism $\eta : \mathrm{I} \to \mathcal{C}$ of cooperads. Denote by $\overline{\mathcal{C}}$ the coaugmentation coideal, that is $\overline{\mathcal{C}} := \mathrm{Coker}(\eta)$.

Let s denote the $\mathbb{S}$-module in graded vector spaces defined by $(0, \mathbf{k}s, 0, \dots)$, where $\mathbf{k}s$ is concentrated in degree -1.

Recall that the cobar construction of a coaugmented dg-cooperad $\mathcal{C}$ (see, for instance, Section 6.5 [LV12]) is given by

$$\Omega(\mathcal{C}) = (\mathbb{T}(s\overline{\mathcal{C}}), d_1 + d_2)$$

where $s\mathcal{C} = s \otimes \mathcal{C}$, d_1 is the derivation extending $d_{\mathcal{C}}$ and d_2 is the derivation extending

$$\delta : s\overline{\mathcal{C}} \xrightarrow{\Delta_s \otimes \Delta_{(1)}} s \otimes s \otimes \overline{\mathcal{C}} \circ_{(1)} \overline{\mathcal{C}} \xrightarrow{\cong} s\overline{\mathcal{C}} \circ_{(1)} s\overline{\mathcal{C}}$$

where $\Delta_s(s) = -s \otimes s$.

We define a quasi-free operad in bicomplexes

$$\mathcal{P}_\infty^{pp} := \left(\mathbb{T}(\mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}})), \partial_1 + \partial_2, \bar{\partial}_1 + \bar{\partial}_2\right)$$

where ∂_1 and $\bar{\partial}_1$ are the derivations extending the differentials of $\mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}})$. The derivations ∂_2 and $\bar{\partial}_2$ are defined as follows. Recall from Section 1.4 the maps of bidegree $(1, 0)$

$$\mathfrak{s}_n : \mathsf{L}_n \longrightarrow \mathsf{L}_{n+1}$$

defined for $n \leq 0$. Then ∂_2 is the derivation extending the following map:

$$\mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}}) \xrightarrow{\mathfrak{s} \otimes \delta} \mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}} \circ_{(1)} s\overline{\mathcal{P}^{\mathsf{i}}}) \xrightarrow{\psi_{s\mathcal{P}^{\mathsf{i}}, s\mathcal{P}^{\mathsf{i}}}} \mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}}) \circ_{(1)} \mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}})$$

where $\mathfrak{s}$ is induced by the maps $\mathfrak{s}_n$, δ is the generating map for the derivation of the cobar construction given above, and $\psi_{s\mathcal{P}^{\mathsf{i}}, s\mathcal{P}^{\mathsf{i}}}$ is the colax structure of the inflation functor, see Proposition 2.1.5.

Explicitly, let μ be an element in $\mathcal{P}^{\mathsf{i}}$ and denote $\mu^{p,q} := (p, q)_{|\mu|+1} \otimes s\mu \in \mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}})$. We have:

$$(\partial_1 + \partial_2)(\mu^{p,q}) = q\mu^{p+1,q} + \sum_i \sum_{\substack{p_1+p_2=p+1 \\ q_1+q_1=q}} (-1)^{\varepsilon+|\mu_i'|} (\mu_i'^{p_1,q_1} \circ_{e_i} \mu_i''^{p_2,q_2}) \sigma_i$$

where $\varepsilon = p + q + |\mu| + 1 + (p_1 + q_1)(p_2 + q_2 + |\mu_i''| + 1)$. Similarly, we use the maps $\bar{\mathfrak{s}}$ to construct $\bar{\partial}_2$ as the derivation extending

$$\mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}}) \xrightarrow{\bar{\mathfrak{s}} \otimes \delta} \mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}} \circ_{(1)} s\overline{\mathcal{P}^{\mathsf{i}}}) \xrightarrow{\psi_{s\mathcal{P}^{\mathsf{i}}, s\mathcal{P}^{\mathsf{i}}}} \mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}}) \circ_{(1)} \mathrm{Inf}(s\overline{\mathcal{P}^{\mathsf{i}}}).$$

In this case, we have:

$$(\bar{\partial}_1 + \bar{\partial}_2)(\mu^{p,q}) = -p\mu^{p,q+1} + \sum_i \sum_{\substack{p_1+p_2=p \\ q_1+q_1=q+1}} (-1)^{\varepsilon+|\mu_i'|} (\mu_i'^{p_1,q_1} \circ_{e_i} \mu_i''^{p_2,q_2}) \sigma_i.$$

Proposition 2.6.8. *The tuple $(\mathcal{P}_\infty^{pp}, \partial, \bar\partial)$ is a bicomplex:*

$$\partial^2 = \bar\partial^2 = \partial\bar\partial + \bar\partial\partial = 0.$$

Proof. We will prove that $\partial^2 = 0$. The other two identities follow by analogous arguments. Recall that the colax structure of the inflation functor involves the maps introduced in Section 1.4

$$\iota_{k_1,\dots,k_n} : \llcorner_{k_1+\dots+k_n} \longrightarrow \llcorner_{k_1} \otimes \cdots \otimes \llcorner_{k_n}.$$

For the computations below, we write $d_i = |s\mu_i|$.

$$\begin{aligned}
\partial\partial((p,q)_n \otimes s\mu) = {} & \sum (-1)^{d_1+1}\, (\iota_{d_1,d_2}\mathfrak{s}\partial_\llcorner(p,q)_n) \otimes s\mu_{(1)} \circ_i s\mu_{(2)} \\
& + \sum (-1)^{d_1+1}\, (\partial_\llcorner \iota_{d_1,d_2}\mathfrak{s}(p,q)_n) \otimes s\mu_{(1)} \circ_i s\mu_{(2)} \\
& + \sum (-1)^{d_2+1}\, ((\iota_{d_1,d_2}\mathfrak{s} \otimes \mathrm{id})\iota_{d_1+d_2+1,d_3}\mathfrak{s}(p,q)_n) \otimes s\mu_{(1)} \circ_i s\mu_{(2)} \circ_j s\mu_{(3)} \\
& + \sum (-1)^{d_1+d_2} (\mathrm{id} \otimes \iota_{d_2,d_3}\mathfrak{s})\iota_{d_1,d_2+d_3+1}\mathfrak{s}(p,q)_n \otimes s\mu_{(1)} \circ_i s\mu_{(2)} \circ_j s\mu_{(3)}.
\end{aligned}$$

The two first summands vanish because $\iota_{k,\ell}$ are maps of bicomplexes and by Remark 1.4.3. The two last summands vanish by the relations satisfied by $\iota_{k,\ell}$ and $\mathfrak{s}$, Proposition 1.4.6. ☐

Remark 2.6.9. There is a naive cobar construction for a coaugmented cooperad $\mathcal{C}$ in $\mathrm{BiCo}_{\mathbf{k}}$ using the shift $\neg : \mathrm{BiCo}_{\mathbf{k}} \to \mathrm{BiCo}_{\mathbf{k}}$, defined in Section 1.1, that mimics the cobar construction in the dg case. However, since the shift does not preserve minimality of bicomplexes, we will typically obtain an operad $\Omega\mathcal{C}$ which is not minimal as an operad in $\mathrm{BiCo}_{\mathbf{k}}$, see Definition 2.4.13. The idea of the construction presented here is shifting before and then apply the inflation functor, so that minimality is ensured.

Theorem 2.6.10 (Minimal model of a Koszul operad)**.** *The operad $\mathcal{P}_\infty^{pp}$ is the pluripotential minimal model of $\mathcal{P}$.*

Proof. We first show that $\mathcal{P}_\infty^{pp}$ is a minimal operad, see Section 2.4. We need to show that it is cofibrant and $\partial\bar\partial$ decomposes, that is:

$$\partial\bar\partial(\mathrm{Inf}(s\overline{\mathcal{P}^{\text{¡}}})) \subseteq (\mathcal{P}_\infty^{pp})^{(\geq 2)}.$$

Note first that $\mathcal{P}_\infty^{pp}$ is cofibrant since it is the colimit of a sequence of principal extensions starting from the initial operad

$$\mathrm{I} \to \mathbb{T}(\mathrm{I} \oplus V_2) \to \cdots \to \mathbb{T}(\mathcal{P}_{n-1} \oplus V_n) \to \cdots \to \mathrm{colim}_n \mathcal{P}_n = \mathcal{P}_\infty^{pp}$$

where the extensions are given by

$$V_{2n} = H_{\mathcal{BC}}(\mathrm{Inf}(s\overline{\mathcal{P}^{\text{¡}}}(n))) \text{ and } V_{2n+1} = H_{\mathcal{A}}(\mathrm{Inf}(s\overline{\mathcal{P}^{\text{¡}}}(n))).$$

By construction, ∂_2 and $\bar\partial_2$ decompose. Note that ∂_1 and $\bar\partial_1$ do not decompose in general but in each arity, $\mathrm{Inf}(s\overline{\mathcal{P}^{\text{¡}}})$ is a minimal bicomplex, and so $\partial_1\bar\partial_1 = 0$. This shows that $\partial\bar\partial$ decomposes and so the operad is minimal.

By writing $\mathcal{P} = \mathcal{P}(E, R)$, define $\alpha\colon \mathcal{P}_\infty^{pp} \to \mathcal{P}$ as the map extending

$$\mathrm{Inf}(s\overline{\mathcal{P}^{¡}}) = \mathrm{Inf}(s\mathcal{C}(s^{-1}E, s^{-2}R)) \twoheadrightarrow \mathrm{Inf}(E) = E \hookrightarrow \mathcal{P}.$$

It only remains to show that α is a pluripotential weak equivalence. Since all bicomplexes are bounded, it suffices to show it induces isomorphisms in H_∂ and $H_{\bar\partial}$, see Remark 1.1.9. Note that, by considering $\mathcal{P}^{¡}$ in bidegree $(*, 0)$, we may view it as a sub-cooperad of $\mathcal{P}^{¡}_{pp}$. Moreover, we have $H_\partial(\mathcal{P}^{¡}_{pp}(n)) = \mathcal{P}^{¡}(n)$ so this inclusion of cooperads is an H_∂-isomorphism. This allows us to identify $\mathcal{P}_\infty$ as the suboperad of $\mathcal{P}_\infty^{pp}$ generated by $H_\partial(\mathrm{Inf}(s\overline{\mathcal{P}^{¡}}))$. In particular, we have an inclusion $\mathcal{P}_\infty \hookrightarrow \mathcal{P}_\infty^{pp}$, which we claim is an H_∂-isomorphism. Indeed, $\mathcal{P}_\infty^{pp}$ is weight-graded since it is a free operad. The differential ∂ of $\mathcal{P}_\infty^{pp}$ splits into the two components $\partial = \partial_1 + \partial_2$ of weight 0 and 1, respectively. So we get that $\mathcal{P}_\infty^{pp}$ is a bicomplex with total differential ∂. In a similar way, $\mathcal{P}_\infty$ is a bicomplex with total differential $\partial = \partial_2$. Moreover, the inclusion $\mathcal{P}_\infty \hookrightarrow \mathcal{P}_\infty^{pp}$ is a map of bicomplexes with respect to the weight-grading and it is a quasi-isomorphism with respect to the weight 0 component. Therefore, a standard spectral sequence argument shows that $\mathcal{P}_\infty \hookrightarrow \mathcal{P}_\infty^{pp}$ is an H_∂-isomorphism.

The diagram

$$\begin{tikzcd} \mathcal{P}_\infty^{pp} \arrow[dr, "\alpha"] & \\ \mathcal{P}_\infty \arrow[u, "\sim", "\partial"'] \arrow[r, "\sim", "\partial"'] & \mathcal{P} \end{tikzcd}$$

commutes and so α is an H_∂-isomorphism. An analogous argument embedding $\mathcal{P}^{¡}$ in bidegree $(0, *)$ gives that α is an $H_{\bar\partial}$-isomorphism. □

Example 2.6.11. Let $\mathcal{P} = As$ (as a non-symmetric operad), then $A_\infty^{pp} = \mathbb{T}(\mathrm{Inf}(sAs^{¡}))$ and the differentials give

$$\partial\mu_n^{p,q} = q\mu_n^{p+1,q} + \sum_{\substack{k+\ell=n+1\\1\le i\le k}} \sum_{\substack{p_1+p_2=p+1\\q_1+q_2=q}} (-1)^{i(\ell+1)+n+\varepsilon}\mu_k^{p_1,q_1} \circ_i \mu_\ell^{p_2,q_2}$$

and

$$\bar\partial\mu_n^{p,q} = -p\mu_n^{p,q+1} + \sum_{\substack{k+\ell=n+1\\1\le i\le k}} \sum_{\substack{p_1+p_2=p\\q_1+q_2=q+1}} (-1)^{i(\ell+1)+n+\varepsilon}\mu_k^{p_1,q_1} \circ_i \mu_\ell^{p_2,q_2},$$

where $\varepsilon = p+q+n+(p_1+q_1)(p_2+q_2+\ell)$ and we have set $\mu_n^{p,q} := (p,q)_{2-n} \otimes s\mu_n$ for $\mu_n \in \mathrm{As}^{¡}(n)$. Algebras over this operad are precisely the pluripotential A_∞-algebras that will be introduced in Section 2.9.

Example 2.6.12. If $\mathcal{P} = Lie$, we obtain $L_\infty^{pp} = \mathbb{T}(\mathrm{Inf}(sLie^{¡}))$ and the differentials give

$$\partial\ell_n^{p,q} = q\ell_n^{p+1,q} + \sum_{r+s=n+1} \sum_{\sigma\in \mathrm{Sh}_{r,s}^{-1}} \sum_{\substack{p_1+p_2=p+1\\q_1+q_2=q}} (-1)^{r(s+1)+\varepsilon}\,\mathrm{sgn}(\sigma)(\ell_r^{p_1,q_1} \circ_1 \ell_s^{p_2,q_2})\sigma$$

and

$$\bar\partial\ell_n^{p,q} = -p\ell_n^{p,q+1} + \sum_{r+s=n+1} \sum_{\sigma\in \mathrm{Sh}_{r,s}^{-1}} \sum_{\substack{p_1+p_2=p\\q_1+q_2=q+1}} (-1)^{r(s+1)+\varepsilon}\,\mathrm{sgn}(\sigma)(\ell_r^{p_1,q_1} \circ_1 \ell_s^{p_2,q_2})\sigma,$$

where $\varepsilon = p+q+n+(p_1+q_1)(p_2+q_2+s)$. We obtain thus, that a pluripotential L_∞-algebra structure on a bicomplex $(L, \partial, \bar{\partial})$ is a family of operations

$$\ell_n^{p,q} : L^{\otimes n} \longrightarrow L, \qquad n = 2, 3, \ldots,$$

of bidegrees $1-n \leq p+q \leq 2-n$, such that each $\ell_n^{p,q}$ is anti-symmetric and such that

$$[\partial, \ell_n^{p,q}] = q\ell_n^{p+1,q} + \sum_{r+s=n+1} \sum_{\sigma \in \mathrm{Sh}_{r,s}^{-1}} \sum_{\substack{p_1+p_2=p+1 \\ q_1+q_2=q}} \mathrm{sgn}(\sigma)\,(-1)^{r(s+1)+\varepsilon}\, \ell_r^{p_1,q_1} \circ \big(\ell_s^{p_2,q_2} \otimes \mathrm{id}^{\otimes(r-1)}\big)\sigma$$

and

$$[\bar{\partial}, \ell_n^{p,q}] = -p\ell_n^{p,q+1} + \sum_{r+s=n+1} \sum_{\sigma \in \mathrm{Sh}_{r,s}^{-1}} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q+1}} \mathrm{sgn}(\sigma)\,(-1)^{r(s+1)+\varepsilon}\, \ell_r^{p_1,q_1} \circ \big(\ell_s^{p_2,q_2} \otimes \mathrm{id}^{\otimes(r-1)}\big)\sigma$$

hold.

2.7 Bar-Cobar constructions for $\mathcal{P}$-algebras

In this section, we introduce the bar and cobar constructions for $\mathcal{P}$-algebras and give an alternative description of $\mathcal{P}_\infty^{pp}$-algebras in terms of codifferentials on cofree coalgebras.

Let $\mathcal{P}$ and $\mathcal{C}$ be an operad and a cooperad in bicomplexes. For the sake of simplicity, we will restrict to *connected cooperads*, that is $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = \mathbf{k}$, which is sufficient for our interests. This condition may be removed assuming conilpotency on coalgebras (see Section 5.8.4 in [LV12]).

Using the notion of twisting morphism $\varphi = (\varphi^{10}, \varphi^{01}) : \mathcal{C} \to \mathcal{P}$ introduced in Section 2.5, we will associate a pair of adjoint functors

$$\Omega_\varphi : \mathcal{C}\text{-coalgebras} \rightleftarrows \mathcal{P}\text{-algebras} : \mathrm{B}_\varphi$$

relating algebras and coalgebras in bicomplexes. The construction is analogous to the classical dg setting with the obvious difference introduced by the bicomplex structures. We will follow the construction as explained by Getzler and Jones in [GJ94].

Let $(A, \partial_A, \bar{\partial}_A, \gamma)$ be a $\mathcal{P}$-algebra. Consider the cofree $\mathcal{C}$-coalgebra $\mathcal{C}(A)$ and the induced square zero coderivations

$$\partial_1 := \partial_{\mathcal{C}} \circ \mathrm{id}_A + \mathrm{id}_{\mathcal{C}} \circ' \partial_A \quad \text{and} \quad \bar{\partial}_1 := \bar{\partial}_{\mathcal{C}} \circ \mathrm{id}_A + \mathrm{id}_{\mathcal{C}} \circ' \bar{\partial}_A.$$

For any linear map $\alpha : \mathcal{C} \to \mathcal{P}$, consider the unique coderivation d_α^c extending

$$\mathcal{C}(A) \xrightarrow{\alpha \circ \mathrm{id}} \mathcal{P}(A) \xrightarrow{\gamma} A$$

as explained in Section 2.3. It is explicitly given by

$$d_\alpha^c := (\mathrm{id}_{\mathcal{C}} \circ (\mathrm{id}_A; \gamma))([(\mathrm{id}_{\mathcal{C}} \circ_{(1)} \alpha)\Delta_{(1)}] \circ \mathrm{id}_A).$$

Note that, for any two linear maps $\alpha, \beta : \mathcal{C} \to \mathcal{P}$, the following relation is satisfied

$$d^c_\alpha d^c_\beta = d^c_{\alpha\star\beta},$$

see for instance [GJ94].

Recall from Definition 2.5.2, that an operadic twisting morphism φ is given by two linear maps φ^{10} and φ^{01} which we will use to construct a functor

$$\mathrm{B}_\varphi : \mathcal{P}\text{-Alg} \to \mathcal{C}\text{-Coalg}$$

called the *bar construction with respect to* φ. It is given on objects by

$$\mathrm{B}_\varphi(A) := (\mathcal{C}(A), \partial_1 + d^c_{\varphi^{10}}, \bar{\partial}_1 + d^c_{\varphi^{01}}).$$

By definition, $B_\varphi(A)$ is a $\mathcal{C}$-coalgebra with two additional coderivations. The following proposition shows that these are indeed anti-commuting codifferentials.

Proposition 2.7.1. *The bar construction $B_\varphi(A)$ is a bicomplex:*

$$\begin{cases} (\partial_1 + d_{\varphi^{10}})^2 = 0, \\ (\bar{\partial}_1 + d_{\varphi^{01}})^2 = 0, \ \ \textit{and} \\ (\partial_1 + d_{\varphi^{10}})(\bar{\partial}_1 + d_{\varphi^{01}}) + (\bar{\partial}_1 + d_{\varphi^{01}})(\partial_1 + d_{\varphi^{10}}) = 0. \end{cases}$$

Proof. The proof is parallel to that of Proposition 2.5.6. First, note that, as in the classical case, for any linear map $\alpha : \mathcal{C} \to \mathcal{P}$ we have

$$\partial_1 d^c_\alpha + d^c_\alpha \partial_1 = d^c_{\partial\alpha+\alpha\partial} \text{ and } \bar{\partial}_1 d^c_\alpha + d^c_\alpha \bar{\partial}_1 = d^c_{\bar{\partial}\alpha+\alpha\bar{\partial}}.$$

Since φ is a twisting morphism, the following relations are satisfied

$$[\partial, \varphi^{10}] + \varphi^{10}\star\varphi^{10} = 0, \quad [\bar{\partial}, \varphi^{01}] + \varphi^{01}\star\varphi^{01} = 0 \quad \text{and} \quad [\partial, \varphi^{01}] + \varphi^{01}\star\varphi^{10} + [\bar{\partial}, \varphi^{10}] + \varphi^{10}\star\varphi^{01} = 0.$$

Together with the fact that $d^c_\alpha d^c_\beta = d^c_{\alpha\star\beta}$ imply

$$(\partial_1 + d^c_{\varphi^{10}})^2 = 0, \quad (\bar{\partial}_1 + d^c_{\varphi^{01}})^2 = 0,$$

and

$$\begin{aligned} (\partial_1 + d^c_{\varphi^{10}})(\bar{\partial}_1 + d^c_{\varphi^{01}}) + (\bar{\partial}_1 + d^c_{\varphi^{01}})(\partial_1 + d^c_{\varphi^{10}}) = \\ = d^c_{[\partial,\varphi^{01}]+\varphi^{01}\star\varphi^{10}} + d^c_{[\bar{\partial},\varphi^{10}]+\varphi^{10}\star\varphi^{01}} = 0. \end{aligned}$$

□

Let $(C, \Delta, \partial, \bar{\partial})$ be a $\mathcal{C}$-coalgebra. Consider the free $\mathcal{P}$-algebra $\mathcal{P}(C)$ and the induced square zero derivations

$$\partial_1 := \partial_{\mathcal{P}} \circ \mathrm{id}_C + \mathrm{id}_{\mathcal{P}} \circ' \partial_C \quad \text{and} \quad \bar{\partial}_1 := \bar{\partial}_{\mathcal{P}} \circ \mathrm{id}_C + \mathrm{id}_{\mathcal{P}} \circ' \bar{\partial}_C.$$

Let $\alpha : \mathcal{C} \to \mathcal{P}$ be a linear map and consider the unique derivation d^a_ϕ extending the following linear map

$$C \xrightarrow{\Delta} \mathcal{C}(C) \xrightarrow{\alpha \circ \mathrm{id}} \mathcal{P}(C),$$

it is explicitly given by

$$d^a_\alpha := (\gamma_{(1)} \circ \mathrm{id}_C)(\mathrm{id}_\mathcal{P} \circ' [(\alpha \circ \mathrm{id})\Delta]).$$

Again, for any two maps $\alpha, \beta : \mathcal{C} \to \mathcal{P}$, the following relation is satisfied

$$d^a_\alpha d^a_\beta = d^a_{\alpha \star \beta}.$$

Let $\varphi : \mathcal{C} \to \mathcal{P}$ be an operadic twisting morphism. We define the *cobar construction of C with respect to φ* as

$$\Omega_\varphi(C) := (\mathcal{P}(C), -\partial_1 - d^a_{\varphi^{10}}, -\bar{\partial}_1 - d^a_{\varphi^{01}}).$$

The following is proven analogously to Proposition 2.7.1.

Proposition 2.7.2. *The cobar construction $\Omega_\varphi(C)$ is a bicomplex.*

We will now show that the two functors constructed above form an adjunction. Let $\varphi : \mathcal{C} \to \mathcal{P}$ be an operadic twisting morphism, $(C, \Delta, \partial, \bar{\partial})$ a $\mathcal{C}$-coalgebra and $(A, \gamma, \partial, \bar{\partial})$ a $\mathcal{P}$-algebra. We define two unary operations $\star_\varphi$ and $\bar{\star}_\varphi$ in $\underline{\mathrm{Hom}}(C, A)$ of degree $(1, 0)$ and $(0, 1)$, respectively:

$$\star_\varphi(\alpha) : C \xrightarrow{\Delta} \mathcal{C}(C) \xrightarrow{\varphi^{10} \circ \alpha} \mathcal{P}(A) \xrightarrow{\gamma} A$$

and

$$\bar{\star}_\varphi(\alpha) : C \xrightarrow{\Delta} \mathcal{C}(C) \xrightarrow{\varphi^{01} \circ \alpha} \mathcal{P}(A) \xrightarrow{\gamma} A.$$

A map $\alpha : C \to A$ of degree $(0, 0)$ is a *twisting morphism with respect to φ* if it satisfies the following two relations

$$[\partial, \alpha] + \star_\varphi(\alpha) = 0 \quad \text{and} \quad [\bar{\partial}, \alpha] + \bar{\star}_\varphi(\alpha) = 0.$$

Proposition 2.7.3. *Let $\varphi : \mathcal{C} \to \mathcal{P}$ be a twisting morphism. For every $\mathcal{P}$-algebra and every $\mathcal{C}$-coalgebra there exist natural bijections*

$$\mathrm{Hom}_{\mathcal{P}\text{-}Alg}(\Omega_\varphi C, A) \cong \mathrm{Tw}_\varphi(C, A) \cong \mathrm{Hom}_{\mathcal{C}\text{-}Coalg}(C, \mathrm{B}_\varphi A).$$

Proof. Since Ω_φ is a free $\mathcal{P}$-algebra, a map $\tilde{\alpha} : \Omega_\varphi(C) \to A$ is uniquely determined by its restriction to the generators $\alpha : C \to A$. The fact that $\tilde{\alpha}$ commutes with the differentials is equivalent to the commutativity of the following two diagrams

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\partial_C + (\varphi^{10} \circ \mathrm{id})\Delta"'] & A \arrow[d, "\partial_A"] \\ C + \mathcal{P}(C) \arrow[r, "\alpha + \gamma(\mathrm{id}_\mathcal{P} \circ \alpha)"'] & A \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\bar{\partial}_C + (\varphi^{01} \circ \mathrm{id})\Delta"'] & A \arrow[d, "\bar{\partial}_A"] \\ C + \mathcal{P}(C) \arrow[r, "\alpha + \gamma(\mathrm{id}_\mathcal{P} \circ \alpha)"'] & A \end{tikzcd},$$

which is equivalent to the two equations

$$[\partial, \alpha] + \star_\varphi(\alpha) = 0 \quad \text{and} \quad [\bar{\partial}, \alpha] + \bar{\star}_\varphi(\alpha) = 0.$$

On the other hand, since $\mathrm{B}_\varphi(A)$ is cofree, a map $\tilde{\alpha} : C \to \mathrm{B}_\varphi(A)$ is characterized by its restriction to the cogenerators, $\alpha : C \to A$. Then, $\tilde{\alpha}$ commutes with the differentials if and only if the following two diagrams commute

$$\begin{tikzcd} C \arrow[r, "(\mathrm{id}_C \circ \alpha)\Delta"] \arrow[d, "\partial_C"'] & \mathrm{B}_\varphi(A) \arrow[d, "\partial_A + \gamma(\varphi^{10} \circ \mathrm{id})"] \\ C \arrow[r, "\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "(\mathrm{id}_C \circ \alpha)\Delta"] \arrow[d, "\bar{\partial}_C"'] & \mathrm{B}_\varphi(A) \arrow[d, "\bar{\partial}_A + \gamma(\varphi^{01} \circ \mathrm{id})"] \\ C \arrow[r, "\alpha"] & A \end{tikzcd},$$

which gives

$$[\partial, \alpha] + \star_\varphi(\alpha) = 0 \quad \text{and} \quad [\bar{\partial}, \alpha] + \bar{\star}_\varphi(\alpha) = 0. \qquad \square$$

In analogy with the classical case, we can describe $\mathcal{P}_\infty^{pp}$-algebras in terms of codifferentials on a cofree coalgebra, where $\mathcal{P}_\infty^{pp}$ denotes the pluripotential minimal model of a Koszul operad $\mathcal{P}$, see Section 2.6. In this case, we will obtain two anti-commuting codifferentials. This is done analogously to the classical case, by defining a convenient twisting morphism

$$\iota : \mathcal{P}^{\text{¡}}_{pp} \longrightarrow \mathcal{P}_\infty^{pp}$$

and then resourcing to the bar construction. Classically, ι is just defined by suspension. In our case, we will make use of the maps

$$\mathfrak{s}_n : \mathsf{L}_n \to \mathsf{L}_{n+1} \quad \text{and} \quad \bar{\mathfrak{s}}_n : \mathsf{L}_n \to \mathsf{L}_{n+1}$$

of bidegree $(1,0)$ and $(0,1)$, respectively, defined in Section 1.4. Explicitly, we consider the equivariant maps

$$\iota^{10} \colon \mathcal{P}^{\text{¡}}_{pp} \xrightarrow{\mathfrak{s} \otimes s} \mathcal{P}_\infty^{pp} \quad \text{and} \quad \iota^{01} \colon \mathcal{P}^{\text{¡}}_{pp} \xrightarrow{\bar{\mathfrak{s}} \otimes s} \mathcal{P}_\infty^{pp}$$

given by

$$(i,j)_n \otimes \mu \mapsto (-1)^{i+j+n}(i+1,j)_{n+1} \otimes s\mu$$

and

$$(i,j)_n \otimes \mu \mapsto (-1)^{i+j+n}(i,j+1)_{n+1} \otimes s\mu,$$

respectively.

Proposition 2.7.4. *The pair $\iota := (\iota^{10}, \iota^{01})$ defines a twisting morphism.*

Proof. This follows from the relations satisfied by $\mathfrak{s}$, $\bar{\mathfrak{s}}$, and the differentials, as described in Remark 1.4.3, together with the relations involving $\mathfrak{s}$, $\bar{\mathfrak{s}}$, and the maps $\iota_{k,\ell} \colon \mathsf{L}_{k+\ell} \to \mathsf{L}_k \otimes \mathsf{L}_\ell$ given in Proposition 1.4.6. $\square$

The bar construction given in Section 2.7 associated to the twisting morphism ι gives a functor

$$\mathrm{B}_\iota : \mathcal{P}_\infty^{pp}\text{-Alg} \to \mathcal{P}^{\text{¡}}_{pp}\text{-Coalg}.$$

As a consequence, any $\mathcal{P}_\infty^{pp}$-algebra can be understood as two codifferentials on the $\mathcal{P}^{\text{¡}}_{pp}$-cofree coalgebra which anti-commute. Moreover, this viewpoint naturally gives a definition of ∞-morphism in the pluripotential case:

Definition 2.7.5. Let A and B be two $\mathcal{P}_\infty^{pp}$-algebras. A *pluripotential* ∞*-morphism*, $\overset{pp}{\infty}$-morphism for short, $f : A \rightsquigarrow B$ is a coalgebra morphism between the bar constructions

$$f : \mathrm{B}_\iota(A) \to \mathrm{B}_\iota(B).$$

Remark 2.7.6. Any morphism of cofree coalgebras is characterized by its projection to the cogenerators. So a $\overset{pp}{\infty}$-morphism of $\mathcal{P}_\infty^{pp}$-algebras $f : A \rightsquigarrow B$ is equivalently given by a map

$$\bar{f} : \mathcal{P}^{¡}_{pp}(A) \to B$$

satisfying

$$(\delta_1^B + \delta_{\iota_{10}}^B) f = \overline{f}(\partial_1^A + d_{\iota_{10}}^A) \quad \text{and} \quad (\bar{\delta}_1^B + \delta_{\iota_{01}}^B) f = \overline{f}(\bar{\partial}_1^A + d_{\iota_{01}}^A),$$

where $\partial_1 + d_{\iota_{10}}$ and $\bar{\partial}_1 + d_{\iota_{01}}$ are the codifferentials of the bar construction defined in Section 2.7 and $\delta_1^B + \delta_{\iota_{10}}^B$ and $\overline{\delta}_1^B + \delta_{\iota_{01}}^B$ the projection of such codifferentials to the cogenerators.

We call a $\overset{pp}{\infty}$-morphism a $\overset{pp}{\infty}$*-weak equivalence* if the arity 1 component, that is $f_1 : A \to B$, is a pluripotential weak equivalence.

Example 2.7.7. Taking $\mathcal{P} = As$ (as a non symmetric operad) we obtain that a $\overset{pp}{\infty}$-morphism $f : A \rightsquigarrow B$ of A_∞^{pp}-algebras is a family of maps

$$f_n : \llcorner_n \otimes A^{\otimes n} \to B, \quad \text{for } n \geq 1.$$

The map f has to satisfy

$$(\partial_1^B + d_{\iota_{10}}^B) f = f(\partial_1^A + d_{\iota_{10}}^A) \quad \text{and} \quad (\bar{\partial}_1^B + d_{\iota_{01}}^B) f = f(\bar{\partial}_1^A + d_{\iota_{01}}^A).$$

Unraveling the equation we obtain the equations of a $\overset{pp}{\infty}$-morphism in Definition 2.9.11 of Section 2.9.

Let ∞-$\mathcal{P}_\infty^{pp}$-Alg denote the category whose objects are $\mathcal{P}_\infty^{pp}$-algebras and whose morphisms are given by $\overset{pp}{\infty}$-morphisms. The following is automatic.

Proposition 2.7.8. *The functor* B_ι *extends to a fully faithful embedding* B_ι

$$\mathrm{B}_\iota : \infty\text{-}\mathcal{P}_\infty^{pp}\text{-Alg} \to \{\textit{quasi-free } \mathcal{P}^{¡}_{pp}\text{-Coalg}\}.$$

2.8 Homotopy theory of $\mathcal{P}$-algebras

Throughout this section, $\mathcal{P}$ is a quadratic operad in $\mathrm{Vect}_{\mathbf{k}}$ that is Koszul as dg-operad. Let $\mathcal{P}_\infty^{pp}$ denote its pluripotential minimal model given in Section 2.6. Recall that ∞-$\mathcal{P}_\infty^{pp}$-Alg denotes the category of $\mathcal{P}_\infty^{pp}$-algebras with $\overset{pp}{\infty}$-morphisms, and $\mathcal{P}$-Alg the category of $\mathcal{P}$-algebras (in bicomplexes) with strict morphisms. The main result of this section is the equivalence of homotopy categories

$$\mathrm{Ho}(\mathcal{P}\text{-Alg}) \cong \mathrm{Ho}(\infty\text{-}\mathcal{P}_\infty^{pp}\text{-Alg}).$$

Consider the twisting morphism

$$\kappa : \mathcal{P}^{¡}_{pp} \to \mathcal{P}$$

defined in Section 2.6 for (co)operads in bicomplexes. By Proposition 2.7.3, this gives an adjunction

$$\Omega_\kappa : \mathcal{P}^{¡}_{pp}\text{-Coalg} \rightleftarrows \mathcal{P}\text{-Alg} : \mathrm{B}_\kappa$$

between coalgebras and algebras in bicomplexes. The counit of this adjunction is given as follows: for A a $\mathcal{P}$-algebra in bicomplexes, it is the map of algebras extending

$$\mathcal{P}^{¡}_{pp} \circ A \xrightarrow{\varepsilon \circ \mathrm{id}_A} \mathrm{I} \circ A \cong A$$

where ε denotes the counit of the cooperad $\mathcal{P}^{¡}_{pp}$. That is,

$$\varepsilon_\kappa : \Omega_\kappa \mathrm{B}_\kappa A = \mathcal{P} \circ \mathcal{P}^{¡}_{pp} \circ A \xrightarrow{\mathrm{id}_{\mathcal{P}} \circ \varepsilon \circ \mathrm{id}_A} \mathcal{P}(A) \xrightarrow{\gamma_A} A.$$

The unit is given by the map of coalgebras extending the map

$$C \cong \mathrm{I} \circ C \xrightarrow{\eta \circ \mathrm{id}_C} \mathcal{P} \circ C,$$

where η denotes the unit of the operad $\mathcal{P}$. That is,

$$C \xrightarrow{\Delta_C} \mathcal{P}^{¡}_{pp} \circ C \xrightarrow{\mathrm{id}_{\mathcal{P}^{¡}_{pp}} \circ \eta \circ \mathrm{id}_C} \mathcal{P}^{¡}_{pp} \circ \mathcal{P} \circ C = \mathrm{B}_\kappa \Omega_\kappa C.$$

We have

Proposition 2.8.1. *Let $\mathcal{P}$ be a Koszul operad. For every $\mathcal{P}$-algebra A, the counit*

$$\varepsilon_\kappa : \Omega_\kappa \mathrm{B}_\kappa A \xrightarrow{\sim} A$$

is a pluripotential weak equivalence of $\mathcal{P}$-algebras.

Proof. By definition, we have

$$\Omega_\kappa \mathrm{B}_\kappa A := (\mathcal{P} \circ \mathcal{P}^{¡}_{pp} \circ A, \partial_1 + d^c_{\kappa^{10}} + d^a_{\kappa^{10}}, \bar{\partial}_1 + d^c_{\kappa^{01}} + d^a_{\kappa^{01}}),$$

where ∂_1 and $\bar{\partial}_1$ are the induced differentials of A on $\mathcal{P} \circ \mathcal{P}^{¡}_{pp} \circ A$, $d^c_{\kappa^{10}}$ is the induced differential by that of the bar construction

$$\mathcal{P}^{¡}_{pp}(A) \to \mathcal{P}^{¡}_{pp} \circ_{(1)} \mathcal{P}^{¡}_{pp}(A) \xrightarrow{(\mathrm{id}_{\mathcal{C}} \circ_{(1)} \kappa^{10}) \circ \mathrm{id}_A} (\mathcal{P}^{¡}_{pp} \circ_{(1)} \mathcal{P}) \circ A \xrightarrow{(\mathrm{id};\gamma_A)} \mathcal{P}^{¡}_{pp}(A),$$

and $d^a_{\kappa^{10}}$ is the differential extending the following map

$$\mathcal{P}^{¡}_{pp}(A) \to \mathcal{P}^{¡}_{pp} \circ \mathcal{P}^{¡}_{pp}(A) \xrightarrow{\kappa^{10} \circ \mathrm{id} \circ \mathrm{id}} \mathcal{P} \circ \mathcal{P}^{¡}_{pp}(A).$$

The differentials $d^c_{\kappa^{01}}$ and $d^a_{\kappa^{01}}$ are defined in the same manner but using κ^{01}. Since $\mathcal{P}$ and $\mathcal{P}^{¡}_{pp}$ are weight graded, $\Omega_\kappa \mathrm{B}_\kappa A$ is weight graded, and the weight is defined by the sum of the weights of $\mathcal{P}$ and $\mathcal{P}^{¡}_{pp}$. We define a bounded below and exhaustive increasing filtration

$$F_r(\Omega_\kappa \mathrm{B}_\kappa A) := \{\omega \mid \text{weight of } \omega \leq r\}.$$

The differentials of $\Omega_\kappa \mathrm{B}_\kappa A$ preserve the filtration. Indeed, ∂_1 and $\bar{\partial}_1$ are weight preserving as well as $d^a_{\kappa 10}$ and $d^a_{\kappa 01}$. The differentials $d^c_{\kappa 10}$ and $d^c_{\kappa 01}$ are not weight preserving but decrease the weight, in particular we have

$$d^c_{\kappa **}(F_r(\Omega_\kappa \mathrm{B}_\kappa A)) \subseteq F_{r-1}(\Omega_\kappa \mathrm{B}_\kappa A).$$

These observations and the fact that $d^a_{\kappa 10}$ and $d^a_{\kappa 01}$ are constructed as the differentials of the twisted composite products (Section 2.5) imply

$$\bigoplus_{r \geq 0} F_r(\Omega_\kappa \mathrm{B}_\kappa A)/F_{r-1}(\Omega_\kappa \mathrm{B}_\kappa A) = (\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}_{pp})(A).$$

Observe that, since $\mathcal{P}$ is Koszul, by the ABC-Künneth formulae (Proposition 1.1.14), and using the fact that $\mathbf{k}[\mathbb{S}_n]$ is a semisimple ring, we obtain

$$H_{\mathcal{ABC}}\left((\mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}_{pp})(A)\right) \cong \mathrm{I}(H_{\mathcal{ABC}}(A)) \cong H_{\mathcal{ABC}}(A).$$

This implies

$$H_{\mathcal{BC}}(F_r(\Omega_\kappa \mathrm{B}_\kappa A)/F_{r-1}(\Omega_\kappa \mathrm{B}_\kappa A)) = H_{\mathcal{A}}(F_r(\Omega_\kappa \mathrm{B}_\kappa A)/F_{r-1}(\Omega_\kappa \mathrm{B}_\kappa A)) = 0$$

for $r > 0$.

Endow A with the trivial filtration of weight 0. Then, the counit of the adjunction

$$\varepsilon_\kappa : \Omega_\kappa \mathrm{B}_\kappa A \xrightarrow{\gamma_A(\mathrm{id}_{\mathcal{P}} \circ \varepsilon \circ \mathrm{id}_A)} A$$

is a filtered map. For $r = 0$, we have

$$F_0(\Omega_\kappa \mathrm{B}_\kappa A) = \mathrm{I} \circ \mathrm{I} \circ A \cong A \xrightarrow{\cong} A.$$

Assume inductively that, for all $0 \leq s < r$, we have a pluripotential weak equivalence

$$F_s := F_s(\Omega_\kappa \mathrm{B}_\kappa A) \xrightarrow{\sim} A.$$

The map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & F_{r-1} \arrow[r] \arrow[d] & F_r \arrow[r] \arrow[d] & F_r/F_{r-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & A \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

induces a commutative diagram where the rows are exact (corollary 1.12 in [Ste25a])

$$\begin{tikzcd}
\dots \arrow[r] & H^{p-1q-1}_{\mathcal{A}}(F_{r-1}) \arrow[r] \arrow[d, "\cong"] & H^{p-1q-1}_{\mathcal{A}}(F_r) \arrow[r] \arrow[d] & H^{p-1q-1}_{\mathcal{A}}(F_r/F_{r-1}) \arrow[r] \arrow[d] & H^{pq}_{\mathcal{BC}}(F_{r-1}) \arrow[r] \arrow[d, "\cong"] & H^{pq}_{\mathcal{BC}}(F_r) \arrow[r] \arrow[d] & H^{pq}_{\mathcal{BC}}(F_r/F_{r-1}) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^{p-1q-1}_{\mathcal{A}}(A) \arrow[r, "\cong"] & H^{p-1q-1}_{\mathcal{A}}(A) \arrow[r] & 0 \arrow[r] & H^{pq}_{\mathcal{BC}}(A) \arrow[r, "\cong"] & H^{pq}_{\mathcal{BC}}(A) \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

By the observation above, we obtain $F_r(\Omega_\kappa \mathrm{B}_\kappa A) \xrightarrow{\sim} A$. Since the filtration is exhaustive, this implies $\Omega_\kappa \mathrm{B}_\kappa A \xrightarrow{\sim} A$. $\square$

Consider the morphism

$$\varphi : \mathcal{P}_\infty^{pp} \xrightarrow{\sim} \mathcal{P},$$

it induces the inclusion functor

$$\varphi^* : \mathcal{P}\text{-Alg} \hookrightarrow \mathcal{P}_\infty^{pp}\text{-Alg}.$$

Denote by

$$i : \mathcal{P}\text{-Alg} \hookrightarrow \mathcal{P}_\infty^{pp}\text{-Alg} \hookrightarrow \infty\text{-}\mathcal{P}_\infty^{pp}\text{-Alg}$$

the obvious inclusion of categories.

Remark 2.8.2. Recall the twisting morphism

$$\iota : \mathcal{P}_{pp}^{\text{¡}} \longrightarrow \mathcal{P}_\infty^{pp}$$

introduced in Section 2.7. The twisting morphisms κ and ι satisfy $\kappa = \varphi\,\iota$. From this identity, together with the definition of the bar construction (Section 2.7), it follows that

$$\mathrm{B}_\kappa = \mathrm{B}_\iota\, i.$$

The following proposition is proved in exactly the same way as in the dg case, we include it here for completeness.

Proposition 2.8.3. *Let $\mathcal{P}$ be a Koszul operad. There is an adjunction*

$$\Omega_\kappa \mathrm{B}_\iota : \infty\text{-}\mathcal{P}_\infty^{pp}\text{-Alg} \rightleftharpoons \mathcal{P}\text{-Alg} : i$$

Proof. Let A be a $\mathcal{P}_\infty^{pp}$-algebra, consider its bar construction $\mathrm{B}_\iota(A)$. We have the following morphism given by the unit of the adjunction associated to κ

$$\mathrm{B}_\iota A \to \mathrm{B}_\kappa \Omega_\kappa (B_\iota A).$$

By the previous remark, we obtain a morphism

$$\mathrm{B}_\iota A \to \mathrm{B}_\iota (i \Omega_\kappa (B_\iota A)),$$

which gives a $\overset{pp}{\infty}$-morphism by Proposition 2.7.8

$$\eta(A) : A \rightsquigarrow i\Omega_\kappa(\mathrm{B}_\iota A),$$

which induces a natural transformation $\eta : \mathrm{id} \to i\Omega_\kappa \mathrm{B}_\iota$.

Let A be a $\mathcal{P}$-algebra, we have the following morphism of $\mathcal{P}$-algebras

$$\varepsilon_\kappa(A) : \Omega_\kappa \mathrm{B}_\iota i(A) = \Omega_\kappa \mathrm{B}_\kappa(A) \to A,$$

which induces a natural transformation of functors $\varepsilon \colon \Omega_\kappa \mathrm{B}_\iota i \to \mathrm{id}$.

Let A be a $\mathcal{P}_\infty^{pp}$-algebra, the composite $\varepsilon(\Omega_\kappa \mathrm{B}_\iota) \circ \Omega_\kappa \mathrm{B}_\iota \eta$ is equal to

$$\Omega_\kappa \mathrm{B}_\iota(A) \xrightarrow{\Omega_\kappa(\eta_\kappa(B_\iota A))} \Omega_\kappa \mathrm{B}_\kappa \Omega_\kappa \mathrm{B}_\iota(A) \xrightarrow{\varepsilon_\kappa(\Omega_\kappa(\mathrm{B}_\iota A))} \Omega_\kappa \mathrm{B}_\iota(A),$$

which is the identity by the adjunction associated to κ. On the other hand, the composite $i(\varepsilon(A)) \circ \eta(i(A))$ is a $\overset{pp}{\infty}$-morphism, whose image under B_ι is given by

$$\mathrm{B}_\kappa A \xrightarrow{\eta_\kappa(\mathrm{B}_\kappa)} \mathrm{B}_\kappa \Omega_\kappa \mathrm{B}_\kappa A \xrightarrow{\mathrm{B}_\kappa(\varepsilon_\kappa)} \mathrm{B}_\kappa A,$$

which is the identity again by the adjunction associated to κ. □

Theorem 2.8.4 (Rectification). *Let $\mathcal{P}$ be a Koszul operad. There is a natural weak equivalence*

$$\mathrm{id}_{\infty\text{-}\mathcal{P}_\infty^{pp}\text{-Alg}} \to \Omega_\kappa \mathrm{B}_\iota.$$

Proof. To any $\mathcal{P}_\infty^{pp}$-algebra A, we apply the unit of the adjunction from Proposition 2.8.3, namely

$$A \rightsquigarrow i\Omega_\kappa \mathrm{B}_\iota A = \Omega_\kappa \mathrm{B}_\iota A.$$

Recall that this $\overset{pp}{\infty}$-morphism is defined by applying the unit of the adjunction associated to κ to $\mathrm{B}_\iota A$. To show that this map is a $\overset{pp}{\infty}$-weak equivalence, we need to prove that its first component is a pluripotential weak equivalence.

The first component is given by the composite:

$$A \xrightarrow{\cong} \mathrm{I} \circ \mathrm{I} \circ A \hookrightarrow \mathcal{P} \circ \mathcal{P}^{\text{!`}}_{pp} \circ A.$$

We equip $\Omega_\kappa \mathrm{B}_\iota A$ with a filtration by the total weight in $\mathcal{P} \circ \mathcal{P}^{\text{!`}}_{pp}$. As in Proposition 2.8.1, the associated graded object is

$$\bigoplus_{r \geq 0} F^r(\Omega_\kappa \mathrm{B}_\iota A)/F^{r-1}(\Omega_\kappa \mathrm{B}_\iota A) = \mathcal{P} \circ_\kappa \mathcal{P}^{\text{!`}}_{pp} \circ A.$$

By an argument analogous to that in Proposition 2.8.1, we conclude that the morphism $A \rightsquigarrow \Omega_\kappa \mathrm{B}_\iota A$ is indeed a $\overset{pp}{\infty}$-weak equivalence. □

Proposition 2.8.5. *The rectification functor*

$$\Omega_\kappa \mathrm{B}_\iota : \infty\text{-}\mathcal{P}_\infty^{pp}\text{-Alg} \to \mathcal{P}\text{-Alg}$$

sends $\overset{pp}{\infty}$-weak equivalences to pluripotential weak equivalences.

Proof. By Theorem 2.8.4, we have the following commutative diagram of $\overset{pp}{\infty}$-morphisms

$$\begin{tikzcd}
\Omega_\kappa \mathrm{B}_\iota(A) \arrow[r, "\Omega_\kappa \mathrm{B}_\iota(f)"] & \Omega_\kappa \mathrm{B}_\iota(A') \\
A \arrow[u, squiggly, "\sim"] \arrow[r, squiggly, "f"] & A'. \arrow[u, squiggly, "\sim"']
\end{tikzcd}$$

We can restrict the diagram to the first components of the $\overset{pp}{\infty}$-morphisms to obtain the following commutative diagram of maps of bicomplexes

$$\begin{tikzcd} \Omega_\kappa \mathrm{B}_\iota(A) \arrow[r, "\Omega_\kappa \mathrm{B}_\iota(f)"] & \Omega_\kappa \mathrm{B}_\iota(A') \\ A \arrow[u, "\sim"] \arrow[r, "f_1"] & A' \arrow[u, "\sim"'] \end{tikzcd}$$

so f is a $\overset{pp}{\infty}$-weak equivalence if and only if $\Omega_\kappa \mathrm{B}_\iota(f)$ is as well. □

Theorem 2.8.6 (Equivalence of homotopy categories)**.** *Let $\mathcal{P}$ be a Koszul operad. We have an equivalence of homotopy categories*

$$\mathrm{Ho}(\mathcal{P}\text{-Alg}) \cong \mathrm{Ho}(\infty\text{-}\mathcal{P}_\infty^{pp}\text{-Alg}).$$

Proof. Consider the adjuction of Proposition 2.8.3

$$\Omega_\kappa \mathrm{B}_\iota : \infty\text{-}\mathcal{P}_\infty^{pp}\text{-Alg} \rightleftharpoons \mathcal{P}\text{-alg} : i.$$

The functor i preserves weak equivalences by definition. The functor $\Omega_\kappa \mathrm{B}_\iota$ also preserves weak equivalences by Proposition 2.8.5. This implies they induce an adjunction at the level of homotopy categories. Moreover, the counit of the adjunction is given by

$$\varepsilon_\kappa : \Omega_\kappa \mathrm{B}_\iota i(A) = \Omega_\kappa \mathrm{B}_\kappa(A) \to A,$$

which is a pluripotential weak equivalence of $\mathcal{P}$-algebras by Propostition 2.8.1. And the unit of the adjunction is a $\overset{pp}{\infty}$-weak equivalence by Theorem 2.8.4. This proves that the unit and the counit induce an isomorphism at the level of homotopy categories. □

2.9 Pluripotential A_∞-algebras

We now introduce the category of pluripotential A_∞-algebras. As we saw in Section 2.6, these arise as algebras over the pluripotential minimal model of the associative operad.

Originally, A_∞-algebras were introduced by Stasheff in [Sta63] in his study of loop spaces and their higher homotopy associativity. We briefly recall the classical definition: an *A_∞-algebra* consists of a cochain complex (A, d) together with a family of linear maps

$$\mu_n : A^{\otimes n} \to A$$

of degree $2 - n$ for every $n \geq 2$, satisfying the following relation:

$$[d, \mu_n] = \sum_{\substack{k+\ell=n+1 \\ 1\leq r\leq k}} (-1)^{r(1+\ell)+n} \mu_k \left(\mathrm{id}^{\otimes r} \otimes \mu_\ell \otimes \mathrm{id}^{\otimes s}\right)$$

where

$$[d, \mu_n] := d\mu_n - \sum_i (-1)^n \mu_n(\mathrm{id}^{\otimes i-1} \otimes d \otimes \mathrm{id}^{n-i}).$$

For low arities we have the following relations:

- For $n = 2$ we have
$$d\mu_2 = \mu_2(d \otimes \mathrm{id}) + \mu_2(\mathrm{id} \otimes d)$$
- For $n = 3$ we have
$$[d, \mu_3] = \mu_2(\mu_2 \otimes \mathrm{id}) - \mu_2(\mathrm{id} \otimes \mu_2).$$

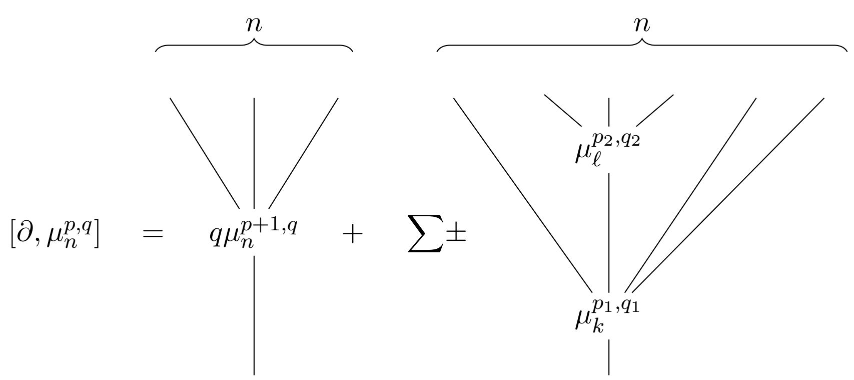
An A_∞-algebra can be viewed as a chain complex equipped with a binary operation that is associative up to a coherent system of higher homotopies. In this sense, a pluripotential A_∞-algebra is analogous, with ordinary homotopy replaced by pluripotential homotopy. Maybe surprisingly, these two definitions are quite different as one can see below.

Definition 2.9.1. A *pluripotential A_∞-algebra*, A_∞^{pp}-algebra for short, consists of a bicomplex $(A, \partial, \bar\partial)$ together n-ary operations of bidegree (p, q)
$$\mu_n^{p,q} : A^{\otimes n} \to A$$
for every $n \geq 2$, where the operations are nonzero only in the following cases:

- when $p + q = 2 - n$ and $p, q \leq 0$,
- when $p + q = 1 - n$ and $p, q \leq 1$.

These operations must satisfy the following relations:
$$[\partial, \mu_n^{p,q}] = q\mu_n^{p+1,q} + \sum_{\substack{k+\ell=n+1\\1\leq i\leq k}} \sum_{\substack{p_1+p_2=p+1\\q_1+q_2=q}} (-1)^{i(\ell+1)+n+\varepsilon} \mu_k^{p_1,q_1}(\mathrm{id}^{\otimes i-1} \otimes \mu_\ell^{p_2,q_2} \otimes \mathrm{id}^{\otimes k-i}), \quad (\mathcal{A}_n^{p,q})$$
and
$$[\bar\partial, \mu_n^{p,q}] = -p\mu_n^{p,q+1} + \sum_{\substack{k+\ell=n+1\\1\leq i\leq k}} \sum_{\substack{p_1+p_2=p\\q_1+q_2=q+1}} (-1)^{i(\ell+1)+n+\varepsilon} \mu_k^{p_1,q_1}(\mathrm{id}^{\otimes i-1} \otimes \mu_\ell^{p_2,q_2} \otimes \mathrm{id}^{\otimes k-i}), \quad (\overline{\mathcal{A}}_n^{p,q})$$
where $\varepsilon = (p_1 + q_1)(\ell + p_2 + q_2) + (p + q + n)$.

Remark 2.9.2. For a fixed n, there are $2n - 3$ operations: $n - 1$ of them have total degree $2 - n$, and $n - 2$ of them have total degree $1 - n$. The relations can be represented as follows:



Here, the sum runs over all compositions of arity n (as in A_∞-algebras) and bidegree $(p+1,q)$. Notice that if $p+q=2-n$, the first tree in the picture vanishes, and all operations in the summation satisfy $p_i+q_i=2-n$ by degree reasons. On the other hand, if $p+q=1-n$, then in each composition in the summation, exactly one operation satisfies $p_i+q_i=1-n$.

Remark 2.9.3. Note that if $(A,\partial,\bar\partial,\{\mu_\bullet^{*,*}\})$ is a A_∞^{pp}-algebra, the tuples $(A,\partial,\{\mu_\bullet^{*,0}\})$ and $(A,\bar\partial,\{\mu_\bullet^{0,*}\})$ are A_∞-algebras.

Example 2.9.4. A bidifferential bigraded algebra, see Example 2.2.7, is an example of a A_∞^{pp}-algebra with $\mu_n^{p,q}=0$ for all $n\geq 3$.

Example 2.9.5. Let us break down the previous definition in the case of low arities.

- For $n=2$, there is a unique bidegree-preserving binary operation

$$\mu_2 : A^{\otimes 2}\to A$$

satisfying

$$\partial\mu_2=\mu_2(\partial,\mathrm{id})+\mu_2(\mathrm{id},\partial)\qquad\text{and}\qquad\bar\partial\mu_2=\mu_2(\bar\partial,\mathrm{id})+\mu_2(\mathrm{id},\bar\partial).$$

This is equivalent to say that both ∂ and $\bar\partial$ are derivations with respect to μ_2.

- For $n=3$, we have three ternary operations

$$\mu_3^{p,q} : A^{\otimes 3}\longrightarrow A,\qquad (p,q)\in\{(-1,0),\,(0,-1),\,(-1,-1)\},$$

satisfying

$$\begin{aligned}
&[\partial,\mu_3^{-1,0}]=\mu_2(\mu_2\otimes\mathrm{id})-\mu_2(\mathrm{id}\otimes\mu_2), && [\bar\partial,\mu_3^{-1,0}]=0,\\
&[\partial,\mu_3^{0,-1}]=0, && [\bar\partial,\mu_3^{0,-1}]=\mu_2(\mu_2\otimes\mathrm{id})-\mu_2(\mathrm{id}\otimes\mu_2),\\
&[\partial,\mu_3^{-1,-1}]=-\,\mu_3^{0,-1}, && [\bar\partial,\mu_3^{-1,-1}]=\mu_3^{-1,0}.
\end{aligned}$$

In particular, these imply

$$[\partial,\,[\bar\partial,\mu_3^{-1,-1}]]\;=\;\mu_2(\mu_2\otimes\mathrm{id})-\mu_2(\mathrm{id}\otimes\mu_2).$$

Thus, μ_2 is associative up to a pluripotential homotopy given by $\mu_3^{-1,-1}$. Note that $\mu_3^{-1,0}$ and $\mu_3^{0,-1}$ are determined by $\mu_3^{-1,-1}$, in fact this holds in general, see the next remark for a detailed explanation.

Remark 2.9.6. Note that

$$[\partial,\mu_n^{p-1,q}]-[\bar\partial,\mu_n^{p,q-1}]=(p+q)\mu_n^{p,q},$$

for any $n\geq 2$ and $p,q\in\mathbb{Z}_{\leq 0}$. This means that, in arity n, the $n-2$ operations of total degree $1-n$ determine the $n-1$ operations of total degree $2-n$.

Definition 2.9.7. Let $(A,\{\mu_\bullet^{*,*}\})$ and $(B,\{\nu_\bullet^{*,*}\})$ be two pluripotential A_∞-algebras. A $\overset{pp}{\infty}$-*morphism* $f:A\rightsquigarrow B$ is a family of linear maps

$$f_n^{p,q} : A^{\otimes n}\to B,\quad n\geq 2,$$

of bidegree (p,q) that are non-zero only in the following cases:

- when $p+q=1-n$ and $p,q\leq 0$,
- when $p+q=-n$ and $p,q\leq -1$

such that

$$
\begin{aligned}
[\partial, f_n^{p,q}] =& q f_n^{p+1,q} - \sum_{i_1+\cdots+i_k=n} \sum_{\substack{p_0+p_{i_1}+\cdots+p_{i_k}=p+1\\ q_0+q_{i_1}+\cdots+q_{i_k}=q}} (-1)^{\theta(i_1,\ldots,i_k)+\varepsilon_1} \nu_k^{p_0,q_0}\left(f_{i_1}^{p_{i_1},q_{i_1}}\otimes\cdots\otimes f_{i_k}^{p_{i_k},q_{i_k}}\right)\\
&- \sum_{\substack{k+\ell=n+1\\ r+s+1=k}} \sum_{\substack{p_1+p_2=p+1\\ q_1+q_2=q}} (-1)^{r(1+\ell)+k+\varepsilon_2} f_k^{p_1,q_1}\left(\mathrm{id}^{\otimes r}\otimes \mu_\ell^{p_2,q_2}\otimes \mathrm{id}^{\otimes s}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
[\bar\partial, f_n^{p,q}] =& - p f_n^{p,q+1} - \sum_{i_1+\cdots+i_k=n} \sum_{\substack{p_0+p_{i_1}+\cdots+p_{i_k}=p\\ q_0+q_{i_1}+\cdots+q_{i_k}=q+1}} (-1)^{\theta(i_1,\ldots,i_k)+\varepsilon_1} \nu_k^{p_0,q_0}\left(f_{i_1}^{p_{i_1},q_{i_1}}\otimes\cdots\otimes f_{i_k}^{p_{i_k},q_{i_k}}\right)\\
&- \sum_{\substack{k+\ell=n+1\\ r+s+1=k}} \sum_{\substack{p_1+p_2=p\\ q_1+q_2=q+1}} (-1)^{r(1+\ell)+k+\varepsilon_2} f_k^{p_1,q_1}\left(\mathrm{id}^{\otimes r}\otimes \mu_\ell^{p_2,q_2}\otimes \mathrm{id}^{\otimes s}\right).
\end{aligned}
$$

Here,

$$\theta := \theta(i_1,\ldots,i_k) := \sum_{1\leq r<s\leq k} i_r(i_s+1),$$

$$\varepsilon_1 := \sum_{j=1}^{k}\left(\sum_{r=0}^{j-1}(p_r+q_r)\right)(p_j+q_j+i_j+1)+p+q+n+1,$$

and

$$\varepsilon_2 = (p_1+q_1)(p_2+q_2+\ell)+p+q+n+1.$$

Remark 2.9.8. For every $n\geq 1$ there are $2n-1$ linear maps, n of total degree $1-n$ and $n-1$ of total degree $-n$.

Example 2.9.9. Again, let us break down the definition of $\overset{pp}{\infty}$-morphisms in the case of low arities.

- For $n=1$, we have a single degree-preserving map

  $$f_1 : A \to B,$$

  such that

  $$[\partial, f_1]=0 \quad \text{and} \quad [\bar\partial, f_1]=0.$$

  In other words, f_1 is a map of bicomplexes.

- For $n=2$, we have three linear maps

  $$f_2^{p,q} : A^{\otimes 2}\to B$$

of bidegrees $(-1,0)$, $(0,-1)$, and $(-1,-1)$ satisfying

$$\begin{aligned}[\partial, f_2^{-1,0}] &= -\mu_2^B(f_1 \otimes f_1) + f_1(\mu_2^A),\\ [\partial, f_2^{0,-1}] &= 0,\\ [\partial, f_2^{-1,-1}] &= -f_2^{0,-1},\end{aligned}$$

and similarly,

$$\begin{aligned}[\bar\partial, f_2^{-1,0}] &= 0,\\ [\bar\partial, f_2^{0,-1}] &= -\mu_2^B(f_1 \otimes f_1) + f_1(\mu_2^A),\\ [\bar\partial, f_2^{-1,-1}] &= f_2^{-1,0}.\end{aligned}$$

In particular, we have

$$[\partial, [\bar\partial, f_2^{-1,-1}]] = -\mu_2^B(f_1 \otimes f_1) + f_1(\mu_2^A).$$

Remark 2.9.10. Note again that

$$[\partial, f_n^{p-1,q}] + [\bar\partial, f_n^{p,q-1}] = (p+q) f_n^{p,q},$$

so, in arity n, the $n-1$ linear maps of total degree $-n$ determine the n linear maps of total degree $1-n$.

Definition 2.9.11. A $\overset{pp}{\infty}$-morphism $f : A \rightsquigarrow B$ is called a *$\overset{pp}{\infty}$-weak equivalence* if f_1 is a pluripotential weak equivalence. We say that f is *strict* if $f_n = 0$ for all $n \geq 2$. The *identity $\overset{pp}{\infty}$-morphism* is the strict morphism with $f_1 = \mathrm{id}_A$.

The composition of two ∞-morphisms $g : A \rightsquigarrow B$ and $f : B \rightsquigarrow C$, is given by

$$(f \circ g)_n = \sum_{i_1+\cdots+i_k=n} \sum_{\substack{p_0+\cdots+p_k=p\\ q_0+\cdots+q_k=q}} (-1)^{\theta(i_1,\ldots,i_k)+\varepsilon} f_k^{p_0,q_0}\left(g_{i_1}^{p_1,q_1} \otimes \cdots \otimes g_{i_k}^{p_k,q_k}\right),$$

where the sign $\theta(i_1,\ldots,i_k)$ is defined as in the $\overset{pp}{\infty}$-morphism relations and

$$\varepsilon := \sum_{j=1}^{k} \left(\sum_{r=0}^{j-1} (p_r + q_r)\right)(p_j + q_j + i_j + 1).$$

This composition is associative, since $\overset{pp}{\infty}$-morphisms are precisely morphisms between the images of the source and the target under the bar construction B_ι, see Section 2.7.

2.10 Homotopy transfer

In this section we show that any bba induces a pluripotential A_∞-algebra structure on any homotopy equivalent bicomplex, with explicit formulas.

Let $(A, \partial, \bar\partial, \mu)$ be a bidifferential bigraded algebra, see Example 2.2.7. For the rest of the section, fix a contraction

$$\begin{tikzcd} (A,\partial,\bar\partial) \arrow["h_{11}"', loop left] \arrow[r, shift left, "f"] & (B,\partial,\bar\partial). \arrow[l, shift left, "g"] \end{tikzcd} \tag{$\mathcal{C}$}$$

Recall from Section 1.2 that the maps satisfy

$$fg = \mathrm{id}_B \quad \text{and} \quad gf - \mathrm{id}_A = [\partial, [\bar\partial, h_{11}]].$$

We define $h_{10} := [\bar\partial, h_{11}]$ and $h_{01} := -[\partial, h_{11}]$, which are chain homotopies with respect to ∂ and $\bar\partial$, respectively:

$$gf - \mathrm{id}_A = [\partial, h_{10}] \quad \text{and} \quad gf - \mathrm{id}_A = [\bar\partial, h_{10}].$$

One can define a binary operation $\nu_2 : B \otimes B \to B$ by the formula

$$\nu_2 := f\mu(g, g).$$

This operation is only associative up to pluripotential homotopy. Indeed, the ternary operation $\nu_3^{\text{-}1,\text{-}1} : B^{\otimes 3} \to B$, defined by the formula

$$\nu_3^{\text{-}1,\text{-}1} := f\mu(h_{11}\mu(g,g), g) - f\mu(g, h_{11}\mu(g,g)),$$

satisfies the relation

$$\nu_2(\nu_2, \mathrm{id}) - \nu_2(\mathrm{id}, \nu_2) = [\partial, [\bar\partial, \nu_3^{\text{-}1,\text{-}1}]].$$

Remark 2.10.1. This operation satisfies $\bar\partial\nu_3^{\text{-}1,\text{-}1} - \nu_3^{\text{-}1,\text{-}1}\bar\partial = \nu_3^{\text{-}1,0}$ and $\partial\nu_3^{\text{-}1,\text{-}1} - \nu_3^{\text{-}1,\text{-}1}\partial = -\nu_3^{0,\text{-}1}$, where

$$\nu_3^{\text{-}1,0} = f\mu(h_{10}\mu(g,g), g) - f\mu(g, h_{10}\mu(g,g)),$$

and

$$\nu_3^{0,\text{-}1} = f\mu(h_{01}\mu(g,g), g) - f\mu(g, h_{01}\mu(g,g)).$$

In general, one can define the following abstract maps associated to a contraction, called $\mathfrak{p}$-*kernels*, which will be used to transfer the algebraic structure of a bba through a minimal contraction.

The associated $\mathfrak{p}$-*kernels* are the linear maps

$$\{\mathfrak{m}_n^{p,q} : A^{\otimes n} \to A\}_{n \geq 2}$$

of bidegree (p, q) where

- $p + q = 2 - n$ and $p, q \leq 0$, or
- $p + q = 1 - n$ and $p, q \leq -1$.

These are defined recursively by

$$\mathfrak{m}_n^{p,q} := \sum_{\substack{k+\ell=n \\ k,\ell\geq 1}} \sum_{\substack{p_1+p_2-\alpha_1-\alpha_2=p \\ q_1+q_2-\beta_1-\beta_2=q}} (-1)^{k(\ell+1)+\varepsilon} \mu(h_{\alpha_1\beta_1}\mathfrak{m}_k^{p_1,q_1} \otimes h_{\alpha_2\beta_2}\mathfrak{m}_\ell^{p_2,q_2}),$$

where $\varepsilon = (\alpha_1+\beta_1)(d_1+k)+(\alpha_1+\beta_1+d_1)(\alpha_2+\beta_2+1)+(\alpha_1+\beta_1+d_1+\alpha_2+\beta_2)(d_2+\ell)$ and $d_x = p_x + q_x$ for $x = 1, 2$. We formally write $h_{11}\mathfrak{m}_1 = \mathrm{id}$, $h_{01}\mathfrak{m}_1 = 0$ and $h_{10}\mathfrak{m}_1 = 0$.

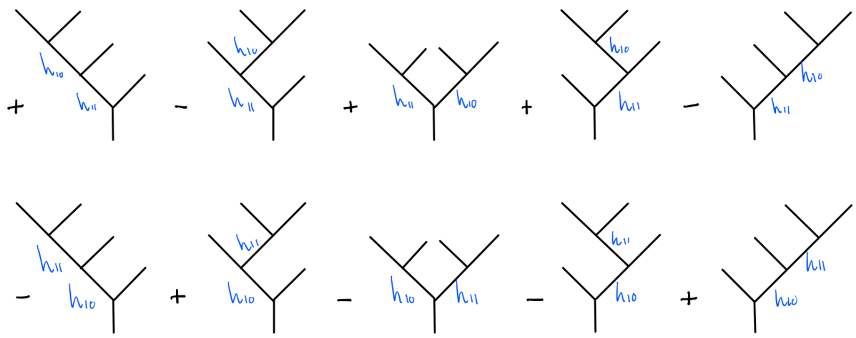
Remark 2.10.2. These maps can be defined as linear combinations of planar binary trees with n leaves, where each internal edge is labeled by h_{10}, h_{01} or h_{11}. Due to degree constraints, only one leaf decorated with h_{11} is permitted per tree. Explicitly, let PBT_n the set of all planar binary trees with n leaves. Then, we have

$$\mathfrak{m}_n^{p,q} = \sum_{t\in \mathrm{PBT}_n} \pm \mathfrak{m}_t^{p,q},$$

where

- if $p+q = 2-n$, $\mathfrak{m}_t^{p,q}$ consists of the sum of $\frac{(n-2)!}{|p|!|q|!}$ copies of t such that each copy is a different way of decorating the internal edges with $|p|$ homotopies h_{10} and $|q|$ homotopies h_{01},
- if $p+q = 1-n$, $\mathfrak{m}_t^{p,q}$ consists of the sum (with signs) of $\frac{(n-2)!}{(|p|-1)!(|q|-1)!}$ copies of t such that each copy is a different way of decorating the internal edges with $|p|-1$ homotopies h_{10}, $|q|-1$ homotopies h_{01} and one homotopy h_{11}.

Example 2.10.3. Below, the operation $\mathfrak{m}_4^{-2,-1}$ is depicted as a sum of planar binary trees with 4 leaves:



Lemma 2.10.4. *Let $(A, \partial, \bar\partial, \mu)$ be a bidifferential bigraded algebra. Given a contraction*

$$h_{11} \circlearrowright (A,\partial,\bar\partial) \underset{g}{\overset{f}{\rightleftarrows}} (B,\partial,\bar\partial),$$

define, for all $n \geq 2$,

$$\nu_n^{p,q} := f\mathfrak{m}_n^{p,q} g^{\otimes n} \quad \textit{and} \quad g_n^{p,q} := \sum_{\substack{i-\alpha=p\\ j-\beta=q}} (-1)^{(\alpha+\beta)(i+j+n)} h_{\alpha\beta}\mathfrak{m}_n^{i,j} g^{\otimes n}.$$

Then, $(B, \partial, \bar\partial, \{\nu_\bullet^{*,*}\})$ is a A_∞^{pp}-algebra, and $g = (g, \{g_\bullet^{*,*}\})$ is a ${}^{pp}_{\infty}$-weak equivalence $g : B \rightsquigarrow A$.

Proof. In the proof of the analogous result in [Mar06], Markl states: "A straightforward but awfully technical induction shows that..." However, the corresponding awfully technical induction is not explicitly presented in the paper. As one might expect, the proof of Lemma 2.10.4 is even worse. But since I'm just a PhD student, I had no choice but to write out the proof, which is given in the remainder of the section. □

Remark 2.10.5. Note that, by the definition of $\mathfrak{m}_n^{p,q}$, we can rewrite $\nu_n^{p,q}$ and $g_n^{p,q}$ as

$$\nu_n^{p,q} = \sum_{k+\ell=n} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} (-1)^{k(\ell+1)+d_1(d_2+\ell+1)} f\mu(g_k^{p_1,q_1} \otimes g_\ell^{p_2,q_2}),$$

and

$$g_n^{p,q} = \sum_{k+\ell=n} \sum_{\substack{p_1+p_2-\alpha=p \\ q_1+q_2-\beta=q}} (-1)^{\varepsilon} h_{\alpha\beta}\mu(g_k^{p_1,q_1} \otimes g_\ell^{p_2,q_2}),$$

where $\varepsilon = k(\ell+1) + d_1(d_2+\ell+1) + (\alpha+\beta)(d_1+d_2+k+\ell)$ and $d_x = p_x + q_x$ for $x = 1, 2$. This expression will be used to prove the unital and commutative versions of the above lemma.

To prove Lemma 2.10.4, we will employ an alternative definition of the $\mathfrak{p}$-kernels for a more compact presentation in the proof. We will use the bicomplexes L_n and the maps

$$\iota_{k_1,\ldots,k_r} : \mathsf{L}_{k_1+\cdots+k_r} \to \mathsf{L}_{k_1} \otimes \cdots \otimes \mathsf{L}_{k_r}$$

defined in Section 1.4.

Remark 2.10.6. Recall from Section 1.4, the maps

$$\mathfrak{s}_1, \bar{\mathfrak{s}}_1 : \mathsf{L} \to \mathbf{k}$$

of bidegree $(1,0)$ and $(0,1)$, respectively. A pluripotential homotopy from $f\colon A \to B$ to $g\colon A \to B$ is equivalent to a bidegree preserving map $\boldsymbol{h}\colon \mathsf{L} \otimes A \to B$ such that

$$[\partial, \boldsymbol{h}] = (f-g)(\mathfrak{s}_1 \otimes \mathrm{id}) \quad \text{ and } \quad [\bar{\partial}, \boldsymbol{h}] = (f-g)(\bar{\mathfrak{s}}_1 \otimes \mathrm{id}).$$

The $\mathfrak{p}$-kernels are equivalent to bidegree preserving linear maps

$$\mathfrak{m}_n \colon \mathsf{L}_{2-n} \otimes A^{\otimes n} \to A,$$

for $n \geq 2$ given by

$$\mathfrak{m}_n \colon = \sum_{k+\ell=n} (-1)^{k(\ell+1)} \mu\big(\boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_k) \otimes \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_\ell)\big)\Delta^h_{k,\ell}.$$

Here,

$$\Delta^h_{k,\ell} := \tau(\iota_{1,2-k,1,2-\ell} \otimes \mathrm{id}^{\otimes n}) \colon \mathsf{L}_{2-n} \otimes A^{\otimes n} \to \mathsf{L} \otimes \mathsf{L}_{2-k} \otimes A^{\otimes k} \otimes \mathsf{L} \otimes \mathsf{L}_{2-\ell} \otimes A^{\otimes \ell},$$

for $k, \ell \geq 1$ with the formal convention that $\mathsf{L}_1 = \urcorner$ and $\boldsymbol{h}\big((\text{-}1,\text{-}1) \otimes \mathfrak{m}_1((1,1)_1 \otimes \_)\big) = \mathrm{id}$ and zero otherwise. The map τ denotes the permutation of tensor factors

$$\tau \colon \mathsf{L} \otimes \mathsf{L}_{2-k} \otimes \mathsf{L} \otimes \mathsf{L}_{2-\ell} \otimes A^{\otimes n} \xrightarrow{\cong} \mathsf{L} \otimes \mathsf{L}_{2-k} \otimes A^{\otimes k} \otimes \mathsf{L} \otimes \mathsf{L}_{2-\ell} \otimes A^{\otimes \ell}.$$

To simplify even more the notation, from now on, we will write

$$\Delta^{\partial,r}_{k,\ell} := \tau_r(\iota_{2-k,2-\ell}\mathfrak{s}_{3-k-\ell}\otimes \mathrm{id}^{\otimes k+\ell-1})\colon \mathsf{L}_{3-k-\ell}\otimes A^{\otimes k+\ell-1}\to \mathsf{L}_{2-k}\otimes A^{\otimes r-1}\otimes \mathsf{L}_{2-\ell}\otimes A^{\otimes \ell+k-r},$$

where $\mathfrak{s}_{3-k-\ell}\colon \mathsf{L}_{3-k-\ell}\to \mathsf{L}_{4-k-\ell}$ is the map defined in Section 1.4 and τ_r is again the obvious permutation of tensor products. Likewise, we define

$$\Delta^{\bar\partial,r}_{k,\ell} := \tau_r(\iota_{2-k,2-\ell}\bar{\mathfrak{s}}_{3-k-\ell}\otimes \mathrm{id}^{\otimes k+\ell-1})\colon \mathsf{L}_{3-k-\ell}\otimes A^{\otimes k+\ell-1}\to \mathsf{L}_{2-k}\otimes A^{\otimes r-1}\otimes \mathsf{L}_{2-\ell}\otimes A^{\otimes \ell+k-r}.$$

Lemma 2.10.7. *The $\mathfrak{p}$-kernels satisfy the following relations*

$$[\partial,\mathfrak{m}_n] = \sum_{\substack{k+\ell=n+1\\ 1\le r\le k\\ k,\ell\ge 2}} (-1)^{r(1+\ell)+n}\mathfrak{m}_k(\mathrm{id}^{\otimes r}\otimes gf\mathfrak{m}_\ell\otimes \mathrm{id}^{\otimes k-r})\Delta^{\partial,r}_{k,\ell} \tag{$\mathcal{M}_n$}$$

and

$$[\bar\partial,\mathfrak{m}_n] = \sum_{\substack{k+\ell=n+1\\ 1\le r\le k\\ k,\ell\ge 2}} (-1)^{r(1+\ell)+n}\mathfrak{m}_k(\mathrm{id}^{\otimes r}\otimes gf\mathfrak{m}_\ell\otimes \mathrm{id}^{\otimes k-r})\Delta^{\bar\partial,r}_{k,\ell} \tag{$\overline{\mathcal{M}}_n$}$$

for every $n\ge 2$.

Proof. We will only prove the relations $\mathcal{M}_n$, the relations $\overline{\mathcal{M}}_n$ are analogous. We have $\mathfrak{m}_2=\mu$ and it satisfies the relation ($\mathcal{M}_2$). Assume inductively that $\mathfrak{m}_m$ satisfies the equation ($\mathcal{M}_m$) for all $m<n$. By definition,

$$\begin{aligned}[\partial,\mathfrak{m}_n] = &\sum_{k+\ell=n}(-1)^{k(\ell+1)}\partial\mu\big(\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_k)\otimes\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_\ell)\big)\Delta^h_{k,\ell}\\ &-\sum_{k+\ell=n}(-1)^{k(\ell+1)}\mu\big(\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_k)\otimes\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_\ell)\big)\Delta^h_{k,\ell}\partial,\end{aligned}$$

where, by abuse of notation, ∂ also denotes the induced differential on the tensor product. Using $\partial\mu=\mu\partial$ and $\partial\Delta^h_{k,\ell}=\Delta^h_{k,\ell}\partial$ yields

$$[\partial,\mathfrak{m}_n] =$$

$$+\sum_{k+\ell=n}(-1)^{k(\ell+1)}\mu\big(\partial\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_k)\otimes\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_\ell)\big)\Delta^h_{k,\ell} \tag{P1}$$

$$+\sum_{k+\ell=n}(-1)^{k(\ell+1)}\mu\big(\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_k)\otimes\partial\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_\ell)\big)\Delta^h_{k,\ell} \tag{P2}$$

$$-\sum_{k+\ell=n}(-1)^{k(\ell+1)}\mu\big(\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_k)\partial\otimes\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_\ell)\big)\Delta^h_{k,\ell} \tag{P3}$$

$$-\sum_{k+\ell=n}(-1)^{k(\ell+1)}\mu\big(\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_k)\otimes\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_\ell)\partial\big)\Delta^h_{k,\ell}. \tag{P4}$$

Applying the induction hypothesis

$$-\mathfrak{m}_k\partial = -\partial\mathfrak{m}_k + \sum_{\substack{k'+\ell'=k+1\\ 1\le r\le k'\\ k',\ell'>1}}(-1)^{r(1+\ell')+k}\mathfrak{m}_{k'}(\mathrm{id}^{\otimes r}\otimes gf\mathfrak{m}_{\ell'}\otimes\mathrm{id}^{\otimes k'-r})\Delta^{\partial,r}_{k',\ell'}$$

for $k \geq 2$ to P3 and P4 gives

$$
\begin{aligned}
\mathrm{P3} = & \\
& - \sum_{\substack{k+\ell=n \\ k>1}} (-1)^{k(\ell+1)} \mu\big(\boldsymbol{h}\partial(\mathrm{id} \otimes \mathfrak{m}_k) \otimes \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_\ell)\big)\Delta^h_{k,\ell} \qquad \text{(P3.1)} \\
& + \sum_{\substack{k+\ell=n \\ k>1}} \sum_{\substack{k'+\ell'=k+1 \\ 1\leq r\leq k' \\ k',\ell'>1}} (-1)^{k(\ell+1)+r(1+\ell')+k} \mu\big(\boldsymbol{h}(\mathrm{id}, \mathfrak{m}_{k'}(\mathrm{id}^{\otimes r}, gf\mathfrak{m}_{\ell'}, \mathrm{id}^{\otimes k'-r})\Delta^{\partial,r}_{k',\ell'}), \boldsymbol{h}(\mathrm{id}, \mathfrak{m}_\ell)\big)\Delta^h_{k,\ell} \\
& \qquad \text{(P3.2)} \\
& - (-1)^n \mu\big(\partial \otimes \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_{n-1})\big)\Delta^h_{1,n-1} \qquad \text{(P3.3)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{P4} = & \\
& - \sum_{\substack{k+\ell=n \\ \ell>1}} (-1)^{k(\ell+1)} \mu\big(\boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_k) \otimes \boldsymbol{h}\partial(\mathrm{id} \otimes \mathfrak{m}_\ell)\big)\Delta^h_{k,\ell} \qquad \text{(P4.1)} \\
& + \sum_{\substack{k+\ell=n \\ \ell>1}} \sum_{\substack{k'+\ell'=\ell+1 \\ 1\leq r\leq k' \\ k',\ell'>1}} (-1)^{k(\ell+1)+r(1+\ell')+\ell} \mu\big(\boldsymbol{h}(\mathrm{id}, \mathfrak{m}_k), \boldsymbol{h}(\mathrm{id}, \mathfrak{m}_{k'}(\mathrm{id}^{\otimes r}, gf\mathfrak{m}_{\ell'}, \mathrm{id}^{\otimes k'-r})\Delta^{\partial,r}_{k',\ell'})\big)\Delta^h_{k,\ell} \\
& \qquad \text{(P4.2)} \\
& - \mu\big(\boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_{n-1}) \otimes \partial\big)\Delta^h_{n-1,1}. \qquad \text{(P4.3)}
\end{aligned}
$$

Now grouping P1+P3.1+P3.3 and P2+P4.1+P4.3 leads to

$$
\begin{aligned}
\mathrm{P1{+}P3.1{+}P3.3{+}P2{+}P4.1{+}P4.3} = & \sum_{\substack{k+\ell=n \\ k>1}} (-1)^{k(\ell+1)} \mu\big((\partial\boldsymbol{h} - \boldsymbol{h}\partial)(\mathrm{id} \otimes \mathfrak{m}_k) \otimes \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_\ell)\big)\Delta^h_{k,\ell} \\
& + \sum_{\substack{k+\ell=n \\ l>1}} (-1)^{k(\ell+1)} \mu\big(\boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_k) \otimes (\partial\boldsymbol{h} - \boldsymbol{h}\partial)(\mathrm{id} \otimes \mathfrak{m}_\ell)\big)\Delta^h_{k,\ell}.
\end{aligned}
$$

From Remark 2.10.6, the identity $[\partial, \boldsymbol{h}] = (gf - \mathrm{id})(\mathfrak{s}_1 \otimes \mathrm{id})$ implies

$$
\begin{aligned}
& \sum_{\substack{k+\ell=n \\ k>1}} (-1)^{k(\ell+1)} \mu\big(gf(\mathfrak{s}_1 \otimes \mathfrak{m}_k) \otimes \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_\ell)\big)\Delta^h_{k,\ell} \\
& - \sum_{\substack{k+\ell=n \\ k>1}} (-1)^{k(\ell+1)} \mu\big((\mathfrak{s}_1 \otimes \mathfrak{m}_k) \otimes \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_\ell)\big)\Delta^h_{k,\ell} \\
& + \sum_{\substack{k+\ell=n \\ \ell>1}} (-1)^{k(\ell+1)} \mu\big(\boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_k) \otimes gf(\mathfrak{s}_1 \otimes \mathfrak{m}_\ell)\big)\Delta^h_{k,\ell} \\
& - \sum_{\substack{k+\ell=n \\ \ell>1}} (-1)^{k(\ell+1)} \mu\big(\boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_k) \otimes (\mathfrak{s}_1 \otimes \mathfrak{m}_\ell)\big)\Delta^h_{k,\ell}.
\end{aligned}
$$

Expanding the definitions of $\mathfrak{m}_k$ and $\mathfrak{m}_\ell$, we obtain

$$\begin{aligned}
&-\sum_{\substack{k+\ell=n\\k>1}}(-1)^{k(\ell+1)}\mu\big((\mathfrak{s}_1\otimes\mathfrak{m}_k)\otimes\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_\ell)\big)\Delta^h_{k,\ell}\\
&-\sum_{\substack{k+\ell=n\\\ell>1}}(-1)^{k(\ell+1)}\mu\big(\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_k)\otimes(\mathfrak{s}_1\otimes\mathfrak{m}_\ell)\big)\Delta^h_{k,\ell}=\\
=&-\sum_{k'+\ell'+\ell=n}(-1)^{\varepsilon_1}\mu\Big(\mu\big(\boldsymbol{h}(\mathrm{id},\mathfrak{m}_{k'}),\boldsymbol{h}(\mathrm{id},\mathfrak{m}_{\ell'})\big),\boldsymbol{h}(\mathrm{id},\mathfrak{m}_\ell)\Big)(\mathfrak{s}_1,\Delta^h_{k',\ell'},\mathrm{id}^{\otimes 2+\ell})\Delta^h_{k'+\ell',\ell}\\
&-\sum_{k+k'+\ell'=n}(-1)^{\varepsilon_2}\mu\Big(\boldsymbol{h}(\mathrm{id},\mathfrak{m}_k),\mu\big(\boldsymbol{h}(\mathrm{id},\mathfrak{m}_{k'}),\boldsymbol{h}(\mathrm{id},\mathfrak{m}_{\ell'})\big)\Big)(\mathrm{id}^{\otimes 2+k},\mathfrak{s}_1,\Delta^h_{k',\ell'})\Delta^h_{k,k'+\ell'}=0,
\end{aligned}$$

where $\varepsilon_1=(k'+\ell')(\ell+1)+k'(\ell'+1)$ and $\varepsilon_2=k(k'+\ell'+1)+k'(\ell'+1)$. Here, we have employed the associativity of μ and the relation

$$(\mathfrak{s}_1,\Delta^h_{k_1,k_2},\mathrm{id}^{\otimes 2+k_3})\Delta^h_{k_1+k_2,k_3}=(-1)^{k_1+1}(\mathrm{id}^{\otimes 2+k_1},\mathfrak{s}_1,\Delta^h_{k_2,k_3})\Delta^h_{k_1,k_2+k_3}.$$

We rewrite

$$\begin{aligned}
&\sum_{\substack{k+\ell=n\\k>1}}(-1)^{k(\ell+1)}\mu\big(gf(\mathfrak{s}_1\otimes\mathfrak{m}_k)\otimes\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_\ell)\big)\Delta^h_{k,\ell}\\
&+\sum_{\substack{k+\ell=n\\\ell>1}}(-1)^{k(\ell+1)}\mu\big(\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_k)\otimes gf(\mathfrak{s}_1\otimes\mathfrak{m}_\ell)\big)\Delta^h_{k,\ell}
\end{aligned}$$

as

$$\begin{aligned}
&\sum_{\substack{k+\ell=n\\k>1}}\mu\big(\mathrm{id}\otimes\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_\ell)\big)\Delta^h_{1,\ell}(\mathrm{id}\otimes gf\mathfrak{m}_k\otimes\mathrm{id}^{\otimes\ell})\Delta^{\partial,1}_{\ell+1,k}\\
&+\sum_{\substack{k+\ell=n\\\ell>1}}(-1)^{k(\ell+1)+k+1}\mu\big(\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_k)\otimes\mathrm{id}\big)\Delta^h_{k,1}(\mathrm{id}^{\otimes k+1}\otimes ip\mathfrak{m}_\ell)\Delta^{\partial,k+1}_{k+1,\ell}.
\end{aligned}\tag{P5}$$

The expression P3.2 becomes

$$\sum_{\substack{k'+\ell'+\ell=n+1\\1\leq r\leq k'\\k',\ell'>1}}(-1)^{\varepsilon_1}\mu\big(\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_{k'})\otimes\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_\ell)\big)\Delta^h_{k',\ell}(\mathrm{id}^{\otimes r}\otimes gf\mathfrak{m}_{\ell'}\otimes\mathrm{id}^{\otimes k'-r+\ell})\Delta^{\partial,r}_{k'+\ell,\ell'},$$

where $\varepsilon_1=r(1+\ell')+k'+\ell+1+(k'+\ell'+1)(\ell+1)+\ell'(\ell+1)+1$.

Similarly, we rewrite P4.2 as

$$\sum_{\substack{k+k'+\ell'=n+1\\1\leq r\leq k'\\k',\ell'>1}}(-1)^{\varepsilon_2}\mu\big(\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_k)\otimes\boldsymbol{h}(\mathrm{id}\otimes\mathfrak{m}_{k'})\big)\Delta^h_{k,k'}(\mathrm{id}^{\otimes k+r}\otimes gf\mathfrak{m}_{\ell'}\otimes\mathrm{id}^{\otimes k'-r})\Delta^{\partial,k+r}_{k+k',\ell'},$$

where $\varepsilon_2=k(k'+\ell')+r(1+\ell')+k'+\ell'+k+1$. Finally, reindexing and summing P5+P3.2+P4.2 completes the proof. □

Remark 2.10.8. A pluripotential A_∞-algebra is equivalently defined as a bicomplex $(A, \partial, \bar{\partial})$ together with n-ary operations

$$\boldsymbol{\mu}_n : \mathsf{L}_{n-2} \otimes A^{\otimes n} \to A$$

for each $n \geq 2$, each of bidegree $(0, 0)$, satisfying the following conditions:

$$[\partial, \boldsymbol{\mu}_n] = \sum_{\substack{k+\ell=n+1 \\ 1\leq r\leq k}} (-1)^{r(1+\ell)+n}\, \boldsymbol{\mu}_k \left(\mathrm{id}^{\otimes r} \otimes \boldsymbol{\mu}_\ell \otimes \mathrm{id}^{\otimes k-r}\right) \Delta^{\partial,r}_{k,\ell},$$

$$[\bar{\partial}, \boldsymbol{\mu}_n] = \sum_{\substack{k+\ell=n+1 \\ 1\leq r\leq k}} (-1)^{r(1+\ell)+n}\, \boldsymbol{\mu}_k \left(\mathrm{id}^{\otimes r} \otimes \boldsymbol{\mu}_\ell \otimes \mathrm{id}^{\otimes k-r}\right) \Delta^{\bar{\partial},r}_{k,\ell}.$$

Lemma 2.10.9. *The $\mathfrak{p}$-kernels induce a A^{pp}_∞-structure on B by the formula*

$$\boldsymbol{\nu}_n = f \circ \mathfrak{m}_n \circ (\mathrm{id} \otimes g^{\otimes n}).$$

Proof. By Remark 2.10.8, the operations $\boldsymbol{\nu}_n$ constitute an A^{pp}_∞-structure if and only if

$$[\partial, \boldsymbol{\nu}_n] - \sum_{\substack{k+\ell=n+1 \\ 1\leq r\leq k}} (-1)^{r(1+\ell)+n}\, \boldsymbol{\nu}_k \left(\mathrm{id}^{\otimes r} \otimes \boldsymbol{\nu}_\ell \otimes \mathrm{id}^{\otimes k-r}\right) \Delta^{\partial,r}_{k,\ell} = 0,$$

and

$$[\bar{\partial}, \boldsymbol{\nu}_n] - \sum_{\substack{k+\ell=n+1 \\ 1\leq r\leq k}} (-1)^{r(1+\ell)+n}\, \boldsymbol{\nu}_k \left(\mathrm{id}^{\otimes r} \otimes \boldsymbol{\nu}_\ell \otimes \mathrm{id}^{\otimes k-r}\right) \Delta^{\bar{\partial},r}_{k,\ell} = 0.$$

Expanding the definition of $\boldsymbol{\nu}_n$ in the first expression, we obtain

$$\begin{aligned} &\partial f \mathfrak{m}_n(\mathrm{id} \otimes g^{\otimes n}) - f\mathfrak{m}_n(\partial \otimes g^{\otimes n}) - \sum_{j=1}^{n} f\mathfrak{m}_n(\mathrm{id} \otimes g^{\otimes j-1} \otimes g\partial \otimes g^{n-j}) \\ - &\sum_{\substack{k+\ell=n+1 \\ 1\leq r\leq k}} (-1)^{r(1+\ell)+n}\, f\mathfrak{m}_k \left(\mathrm{id} \otimes g^{\otimes r-1} \otimes gf\mathfrak{m}_\ell(\mathrm{id} \otimes g^{\otimes \ell}) \otimes g^{\otimes k-r}\right) \Delta^{\partial,r}_{k,\ell} \end{aligned}$$

which gives

$$\begin{aligned} &f\Bigg(\partial\mathfrak{m}_n - \sum_{j=1}^{n+1} \mathfrak{m}_n(\mathrm{id}^{\otimes j-1} \otimes \partial \otimes \mathrm{id}^{\otimes n+1-j}) \\ - &\sum_{\substack{k+\ell=n+1 \\ 1\leq r\leq k \\ k,\ell\geq 2}} (-1)^{r(1+\ell)+n}\, \mathfrak{m}_k \left(\mathrm{id}^{\otimes r} \otimes gf\mathfrak{m}_\ell \otimes \mathrm{id}^{\otimes k-r}\right) \Delta^{\partial,r}_{k,\ell}\Bigg)(\mathrm{id} \otimes g^{\otimes n}) = 0 \end{aligned}$$

by Lemma 2.10.7. Here, we use that g and f are maps of bicomplexes and that

$$(\mathrm{id} \otimes g^{\otimes r} \otimes \mathrm{id} \otimes g^{\otimes \ell+s})\Delta^{\partial,r}_{k,\ell} = \Delta^{\partial,r}_{k,\ell}(\mathrm{id} \otimes g^{\otimes n}).$$

A similar proof shows

$$[\bar{\partial}, \boldsymbol{\nu}_n] - \sum_{\substack{k+\ell=n+1\\1\le r\le k}} (-1)^{r(1+\ell)+n}\, \boldsymbol{\nu}_k \left(\mathrm{id}^{\otimes r}\otimes \boldsymbol{\nu}_\ell \otimes \mathrm{id}^{\otimes k-r}\right) \Delta^{\bar{\partial},r}_{k,\ell} = 0.$$

□

Remark 2.10.10. Let $(A, \boldsymbol{\mu}^A_*)$ and $(B, \boldsymbol{\mu}^B_*)$ be two pluripotential A_∞-algebras. Similarly to Remark 2.10.8, a $\overset{pp}{\infty}$-morphism $f\colon A \rightsquigarrow B$ is equivalently defined as a family of bidegree-preserving maps

$$\boldsymbol{f}_n\colon \mathsf{L}_{n-1}\otimes A^{\otimes n} \to B,\quad n\ge 1,$$

which satisfy the following relations:

$$\begin{aligned}[\partial, \boldsymbol{f}_n] = &- \sum_{i_1+\cdots+i_k=n} (-1)^{\theta(i_1,\ldots,i_k)}\, \boldsymbol{\mu}^B_k\Big(\mathrm{id}\otimes \boldsymbol{f}_{i_1}\otimes\cdots\otimes \boldsymbol{f}_{i_k}\Big)\Delta^{\partial}_{k,i_1+1,\ldots,i_k+1}\\ &- \sum_{\substack{k+\ell=n+1\\1\le r\le k}} (-1)^{r(1+\ell)+n}\, \boldsymbol{f}_k\Big(\mathrm{id}^{\otimes r}\otimes \boldsymbol{\mu}^A_\ell\otimes \mathrm{id}^{\otimes k-r}\Big)\Delta^{\partial,r}_{k+1,\ell}.\end{aligned}$$

$$\begin{aligned}[\bar{\partial}, \boldsymbol{f}_n] = &- \sum_{i_1+\cdots+i_k=n} (-1)^{\theta(i_1,\ldots,i_k)}\, \boldsymbol{\mu}^B_k\Big(\mathrm{id}\otimes \boldsymbol{f}_{i_1}\otimes\cdots\otimes \boldsymbol{f}_{i_k}\Big)\Delta^{\bar{\partial}}_{k,i_1+1,\ldots,i_k+1}\\ &- \sum_{\substack{k+\ell=n+1\\1\le r\le k}} (-1)^{r(1+\ell)+n}\, \boldsymbol{f}_k\Big(\mathrm{id}^{\otimes r}\otimes \boldsymbol{\mu}^A_\ell\otimes \mathrm{id}^{\otimes k-r}\Big)\Delta^{\bar{\partial},r}_{k+1,\ell}.\end{aligned}$$

Here,

$$\Delta^{\partial}_{k,i_1+1,\ldots,i_k+1} := (\tau_{i_1,\ldots,i_k})(\iota_{2\text{-}k,1\text{-}i_1,\ldots,1\text{-}i_k})\mathfrak{s}_{1\text{-}n} : \mathsf{L}_{1\text{-}n}\otimes A^{\otimes n} \to \mathsf{L}_{2\text{-}k}\otimes \mathsf{L}_{1\text{-}i_1}\otimes A^{\otimes i_1}\otimes\cdots\otimes \mathsf{L}_{1\text{-}i_k}\otimes A^{\otimes i_k}$$

and

$$\Delta^{\bar{\partial}}_{k,i_1+1,\ldots,i_k+1} := (\tau_{i_1,\ldots,i_k})(\iota_{2\text{-}k,1\text{-}i_1,\ldots,1\text{-}i_k})\bar{\mathfrak{s}}_{1\text{-}n} : \mathsf{L}_{1\text{-}n}\otimes A^{\otimes n} \to \mathsf{L}_{2\text{-}k}\otimes \mathsf{L}_{1\text{-}i_1}\otimes A^{\otimes i_1}\otimes\cdots\otimes \mathsf{L}_{1\text{-}i_k}\otimes A^{\otimes i_k},$$

where $\tau_{i_1,\ldots,i_k}$ is the isomorphism that permutes the tensor factors. The sign θ is given by

$$\theta(i_1,\ldots,i_k) := \sum_{1\le r<s\le k} i_r(i_s+1).$$

Lemma 2.10.11. *The $\mathfrak{p}$-kernels define a $\overset{pp}{\infty}$-morphism $g : B \rightsquigarrow A$ via the formula*

$$\boldsymbol{g}_n := \boldsymbol{h}(\mathrm{id}\otimes \mathfrak{m}_n)(\iota_{\text{-}1,2\text{-}n}\otimes g^{\otimes n}),$$

for $n\ge 2$ and $\boldsymbol{g}_1 = g$.

Proof. Let us compute

$$\partial \boldsymbol{g}_n - \boldsymbol{g}_n\partial = \partial \boldsymbol{h}(\mathrm{id}\otimes \mathfrak{m}_n)(\iota_{\text{-}1,2\text{-}n}\otimes g^{\otimes n}) - \boldsymbol{h}(\mathrm{id}\otimes \mathfrak{m}_n)(\iota_{\text{-}1,2\text{-}n}\otimes g^{\otimes n})\partial.$$

By Lemma 2.10.7

$$
\begin{aligned}
&\partial \boldsymbol{g}_n - \boldsymbol{g}_n \partial = \\
&\qquad = \partial \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_n)(\iota_{-1,2-n} \otimes g^{\otimes n}) - \boldsymbol{h}\partial(\mathrm{id} \otimes \mathfrak{m}_n)(\iota_{-1,2-n} \otimes g^{\otimes n}) \\
&\qquad\quad + \sum_{\substack{k+\ell=n+1 \\ 1\le r\le k \\ k,\ell\ge 2}} (-1)^{\theta} \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_k(\mathrm{id}^{\otimes r}, gf\mathfrak{m}_\ell, \mathrm{id}^{\otimes k-r}))(\mathrm{id} \otimes \Delta^{\partial,r}_{k,\ell} \otimes \mathrm{id}^{\otimes n})(\iota_{-1,2-n} \otimes g^{\otimes n}),
\end{aligned}
$$

with $\theta = r(1+\ell)+n$. Now we use the relation $[\partial, \boldsymbol{h}] = (gf - \mathrm{id})(\mathfrak{s}_1 \otimes \mathrm{id})$ in Remark 2.10.6 and that

$$(\mathrm{id} \otimes \Delta^{\partial,r}_{k,\ell})\iota_{-1,2-n} = -(\iota_{-1,2-k} \otimes \mathrm{id})\Delta^{\partial,r}_{k+1,\ell}$$

to obtain

$$
\begin{aligned}
&gf(\mathfrak{s}_1 \otimes \mathfrak{m}_n)(\iota_{-1,2-n} \otimes g^{\otimes n}) - (\mathfrak{s}_1 \otimes \mathfrak{m}_n)(\iota_{-1,2-n} \otimes g^{\otimes n}) \\
&- \sum_{\substack{k+\ell=n+1 \\ 1\le r\le k \\ k,\ell\ge 2}} (-1)^{r(1+\ell)+n} \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_k)(\iota_{-1,2-k} \otimes \mathrm{id}^{\otimes k})(\mathrm{id}^{\otimes r} \otimes gf\mathfrak{m}_\ell \otimes \mathrm{id}^{\otimes k-r}))\Delta^{\partial,r}_{k+1,\ell}(\mathrm{id} \otimes g^{\otimes n})
\end{aligned}
$$

which gives

$$
\begin{aligned}
gf(\mathfrak{m}_n)(\mathrm{id} \otimes g^{\otimes n})\Delta^{\partial,1}_{2,n} &- \sum_{k+\ell=n} (-1)^{k(\ell+1)}\, \mu\Big(\boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_k) \otimes \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_\ell)\Delta^{h}_{k,\ell}\Big)\Delta^{\partial,1}_{2,n} \otimes (\mathrm{id} \otimes g^{\otimes n}) \\
&- \sum_{\substack{k+\ell=n+1 \\ 1\le r\le k \\ k,\ell\ge 2}} (-1)^{r(1+\ell)+n} \boldsymbol{g}_k(\mathrm{id}^{\otimes r} \otimes \boldsymbol{\nu}_\ell \otimes \mathrm{id}^{\otimes k-r})\Delta^{\partial}_{k+1,\ell} = \\
&= -\sum_{k+\ell=n} (-1)^{k(\ell+1)}\, \mu\Big(\boldsymbol{g}_k \otimes \boldsymbol{g}_\ell\Big)\Delta^{\partial}_{2,k+1,\ell+1} \\
&- \sum_{\substack{k+\ell=n+1 \\ 1\le r\le k}} (-1)^{r(1+\ell)+n} \boldsymbol{g}_k(\mathrm{id}^{\otimes r} \otimes \boldsymbol{\nu}_\ell \otimes \mathrm{id}^{\otimes k-r})\Delta^{\partial}_{k+1,\ell,r}.
\end{aligned}
$$

A similar computation shows

$$
\begin{aligned}
\bar\partial \boldsymbol{g}_n - \boldsymbol{g}_n \bar\partial = &- \sum_{k+\ell=n} (-1)^{k(\ell+1)}\, \mu\Big(\boldsymbol{g}_k \otimes \boldsymbol{g}_\ell\Big)\Delta^{\bar\partial}_{2,k+1,\ell+1} \\
&- \sum_{\substack{k+\ell=n+1 \\ 1\le r\le k}} (-1)^{r(1+\ell)+n} \boldsymbol{g}_k(\mathrm{id}^{\otimes r} \otimes \boldsymbol{\nu}_\ell \otimes \mathrm{id}^{\otimes k-r})\Delta^{\bar\partial}_{k+1,\ell,r}.
\end{aligned}
$$

□

Lemma 2.10.9 and Lemma 2.10.11 imply the following, which is equivalent to Lemma 2.10.4 by Remarks 2.10.8 and 2.10.10.

Lemma 2.10.12. *Let $(A, \partial, \bar\partial, \mu)$ be a bidifferential bigraded algebra and let*

$$
\begin{tikzcd}
(A, \partial, \bar\partial, \mu) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (B, \partial, \bar\partial) \arrow[l, shift left, "g"]
\end{tikzcd}
$$

be a contraction. The $\mathfrak{p}$*-kernels*

$$\mathfrak{m}_n : \mathsf{L}_{2-n} \otimes A^{\otimes n} \to A$$

given by

$$\mathfrak{m}_n := \sum_{k+\ell=n} (-1)^{k(\ell+1)} \mu(\boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_k) \otimes \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_\ell))\Delta^h_{k,\ell}$$

for $n \geq 2$ *induce*

1. *a* A^{pp}_∞*-structure on* B *via* $\boldsymbol{\nu}_n = f \circ \mathfrak{m}_n \circ (\mathrm{id} \otimes g^{\otimes n})$*, and*
2. *a* $\overset{pp}{\infty}$*-weak equivalence* $g : B \rightsquigarrow A$ *via* $\boldsymbol{g}_n := \boldsymbol{h}(\mathrm{id} \otimes \mathfrak{m}_n)(\iota_{\text{-1,2-}n} \otimes g^{\otimes n})$ *for* $n \geq 2$ *and* $\boldsymbol{g}_1 = g$*.*

We now state the main result of this section, which shows that for every bba A, one can construct a $\overset{pp}{\infty}$-weak equivalence $g\colon H \rightsquigarrow A$ where H is a minimal bicomplex.

Theorem 2.10.13 (Homotopy Transfer Theorem)**.** *Let* $(A, \partial, \bar{\partial}, \mu)$ *be a bidifferential bigraded algebra.*

1. *There is a contraction into a minimal bicomplex*

$$h \circlearrowright (A, \partial, \bar{\partial}) \underset{g}{\overset{f}{\rightleftarrows}} (H, \partial, \bar{\partial}) \ ; \quad \textit{with } \partial\bar{\partial} = 0 \textit{ on } H.$$

2. *There is a* A^{pp}_∞*-structure on* $(H, \partial, \bar{\partial})$ *and a* $\overset{pp}{\infty}$*-weak equivalence* $g : H \rightsquigarrow A$*.*

Proof. This is now a direct consequence of Proposition 1.2.4 together with Lemma 2.10.4. ☐

Remark 2.10.14. Since a bba is, in particular, a bicomplex, we can consider its Bott-Chern and Aeppli cohomologies (see Section 1.1). The product induces an associative algebra structure on the Bott-Chern cohomology and equips the Aeppli cohomology with the structure of a module over the Bott-Chern cohomology.

The bicomplex H of the above theorem is isomorphic to the ABC-cohomology $H_{\mathcal{ABC}}(A) = H_{\mathcal{BC}}(A) \oplus \tilde{H}_{\mathcal{A}}(A)$, see again Section 1.1. Consequently, the induced product does not make the ABC-cohomology into a bba. However, Theorem 2.10.13 provides a way to endow this cohomology with an algebra structure that is not strictly associative, but associative up to homotopy.

If the bba A satisfies the $\partial\bar{\partial}$-property (Definition 1.1.2), then the bicomplex H has trivial differentials. Hence, by Remark 2.9.6, we have $\mu^{p,q}_n = 0$ whenever $p+q = 2-n$. Moreover, all the cohomologies considered in Chapter 1 are isomorphic, the bicomplex H is isomorphic to $H_{\mathcal{BC}}(A)$, and the product μ_2 of Theorem 2.10.13 gives precisely the induced product on cohomology.

2.11 Pluripotential C_∞-algebras

In this section we define the analogues of commutative algebras in the setting of A^{pp}_∞-algebras, i.e. C^{pp}_∞-algebras, and we show that if we start with a commutative bidifferential bigraded algebra, then the formulas for the transferred A^{pp}_∞-structure of Lemma 2.10.4 give a C^{pp}_∞-structure.

To define C^{pp}_∞-structures, we first recall the definition of shuffle permutations: A (i,j)-*shuffle* is an element σ of the symmetric group $\mathbb{S}_{i+j}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(i) \quad \text{and} \quad \sigma(i+1) < \sigma(i+2) < \cdots < \sigma(i+j)$$

hold.

Denote by $\mathrm{Sh}_{i,j}$ the set of (i,j)-shuffles. Define the element $\tau_{i,j} \in \mathbf{k}[\mathbb{S}_{i+j}]$ by

$$\tau_{i,j} = \sum_{\sigma \in \mathrm{Sh}_{i,j}} \mathrm{sign}(\sigma)\sigma.$$

Definition 2.11.1. A *pluripotential C_∞-algebra*, C_∞^{pp}-algebra for short, is a A_∞^{pp}-algebra such that

$$\mu_{i+j}^{p,q} \circ \tau_{i,j} = 0$$

for every $i,j \geq 1$ and every $p,q \in \mathbb{Z}_{\leq 0}$.

Here, $\tau_{i,j}$ acts on $A^{\otimes i+j}$ by permuting the tensor factors. A C_∞^{pp}-algebra is then a A_∞^{pp}-algebra satisfying commutativity relations. Likewise, a $\overset{pp}{\infty}$-morphism of C_∞^{pp}-algebras is required to satisfy the same commutativity relations.

Definition 2.11.2. Let $(A, \{\mu_\bullet^{*,*}\})$ and $(B, \{\nu_\bullet^{*,*}\})$ be two C_∞^{pp}-algebras. A *$\overset{pp}{\infty}$-morphism of C_∞^{pp}-algebras* $f : A \rightsquigarrow B$ is a $\overset{pp}{\infty}$-morphism of A_∞^{pp}-algebras such that

$$f_{i+j}^{p,q} \circ \tau_{i,j} = 0$$

for every $i,j \geq 1$ and every $p,q \in \mathbb{Z}_{\leq 0}$.

We next show that when the initial algebra is commutative, then the transferred structure of Lemma 2.10.4 is, in fact, a C_∞^{pp}-structure.

Lemma 2.11.3. *Let $(A, \partial, \bar\partial, \mu)$ be a commutative bidifferential bigraded algebra. Given a contraction*

$$\begin{tikzcd} (A,\partial,\bar\partial) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (B,\partial,\bar\partial) \arrow[l, shift left, "g"] \end{tikzcd},$$

the formulas defined in Lemma 2.10.4

$$\nu_n^{p,q} := f\mathfrak{m}_n^{p,q} g^{\otimes n} \quad \text{and} \quad g_n^{p,q} := \sum_{\substack{i-\alpha=p \\ j-\beta=q}} (-1)^{(\alpha+\beta)(i+j+n)} h_{\alpha\beta}\mathfrak{m}_n^{i,j} g^{\otimes n},$$

for all $n \geq 2$, define a C_∞^{pp}-structure on B and a $\overset{pp}{\infty}$-weak equivalence of C_∞^{pp}-algebras $g\colon B \rightsquigarrow A$.

Proof. The proof is that of Theorem 12 of [CG08] for C_∞-algebras in the dg setting. We reproduce it here for the sake of completeness. By Lemma 2.10.4, $\{\nu_\bullet^{*,*}\}$ defines a A_∞^{pp}-algebra structure on B and $g = (g, \{g_\bullet^{*,*}\})$ defines a $\overset{pp}{\infty}$-morphism. It just remains to prove the symmetric conditions

$$\nu_{i+j}^{p,q} \circ \tau_{i,j} = 0, \quad \text{and} \quad g_{i+j}^{p,q} \circ \tau_{i,j} = 0.$$

First of all, recall that, by Remark 2.10.5, we can rewrite $\nu_n^{p,q}$ and $g_n^{p,q}$ as

$$\nu_n^{p,q} = \sum_{k+\ell=n} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} (-1)^{k(\ell+1)+d_1(d_2+\ell+1)} f\mu(g_k^{p_1,q_1} \otimes g_\ell^{p_2,q_2}),$$

and

$$g_n^{p,q} = \sum_{k+\ell=n} \sum_{\substack{p_1+p_2-\alpha=p \\ q_1+q_2-\beta=q}} (-1)^{k(\ell+1)+d_1(d_2+\ell+1)+(\alpha+\beta)(d_1+d_2+k+\ell)} h_{\alpha\beta}\mu(g_k^{p_1,q_1} \otimes g_\ell^{p_2,q_2}),$$

where $d_x = p_x + q_x$ for $x = 1, 2$.

Let $\nabla_2 : T(B) \to T^2(TB)$ be the map given by

$$a_1 \otimes \cdots \otimes a_n \mapsto \sum_{k+\ell=n} (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_{k+\ell}).$$

Define the shuffle product

$$\sqcup\!\sqcup : T(B) \otimes T(B) \to T(B)$$

by

$$(a_1, \ldots, a_i) \otimes (a_{i+1}, \ldots, a_{i+j}) \mapsto \tau_{i,j}(a_1, \ldots, a_i, a_{i+1}, \ldots, a_{i+j}).$$

By Lemma 13 in [CG08], we have

$$\nabla_2(TB \sqcup\!\sqcup TB) \subset \tau_{1,1}(T^2(TB)) \ \oplus \ TB \ \otimes \ (TB \sqcup\!\sqcup TB) \ \oplus \ (TB \sqcup\!\sqcup TB) \ \otimes \ TB.$$

We will show by induction on $n \geq 2$ that $\nu_n^{p,q}$ and $g_n^{p,q}$ vanish on $TB \sqcup\!\sqcup TB$, which implies $\nu_{i+j}^{p,q}\tau_{i,j} = 0$ and $g_{i+j}^{p,q}\tau_{i,j} = 0$ for any $i, j \geq 1$ and any $p, q \in \mathbb{Z}_{\leq 0}$.

On the one hand, by induction hypothesis, the map

$$\sum_{k+\ell=n} \sum_{\substack{p_1+p_2=i \\ q_1+q_2=j}} g_k^{p_1,q_1} \otimes g_\ell^{p_2,q_2}$$

annihilates

$$TB \ \otimes \ (TB \sqcup\!\sqcup TB) \ \oplus \ (TB \sqcup\!\sqcup TB) \ \otimes \ TB.$$

On the other hand,

$$\sum_{k+\ell=n} \sum_{\substack{p_1+p_2=i \\ q_1+q_2=j}} g_k^{p_1,q_1} \otimes g_\ell^{p_2,q_2}$$

takes the subspace $\tau_{1,1}T^2(TB)$ to $\tau_{1,1}(T^2(A))$, but this is annihilated by μ, and we conclude that $\nu_n^{p,q}(TB \sqcup\!\sqcup TB)$ and $g_n^{p,q}(TB \sqcup\!\sqcup TB)$ vanish. □

As a consequence, we obtain the following commutative version of Theorem 2.10.13.

Theorem 2.11.4. *Let $(A, \partial, \bar{\partial}, \mu)$ be a commutative bidifferential bigraded algebra. There exists a minimal bicomplex H endowed with a C_∞^{pp}-structure and a $\overset{pp}{\infty}$-weak equivalence of C_∞^{pp}-algebras $g\colon H \rightsquigarrow A$.*

2.12 Unital case

In geometric contexts, the algebras that arise are typically unital. For instance, the de Rham algebra of a smooth manifold. In this section, we treat the homotopy transfer theorem in the unital case.

Definition 2.12.1. A *unital A_∞^{pp}-algebra* is a A_∞^{pp}-algebra $(A, \partial, \bar{\partial}, \{\mu_\bullet^{*,*}\})$ together with a strict morphism of algebras $u : \mathbf{k} \to A$ such that

$$\mu_2(u \otimes \mathrm{id}) = \mu_2(\mathrm{id} \otimes u) = \mathrm{id},$$

and

$$\mu_n^{p,q}(\mathrm{id}^{\otimes(i-1)} \otimes u \otimes \mathrm{id}^{\otimes(n-i)}) = 0$$

for all $n \geq 3$, $1 - n \leq p + q \leq 2 - n$, and $1 \leq i \leq n$.

In other words, a unital A_∞^{pp}-algebra is a A_∞^{pp}-algebra endowed with an element 1_A (the image of $1_{\mathbf{k}}$ under u) of degree $(0, 0)$ such that:

$$\partial(1_A) = \bar{\partial}(1_A) = 0, \quad \mu_2(1_A, a) = \mu_2(a, 1_A) = a \quad \text{for all } a \in A,$$

and for all $n \geq 3$ and all $a_1, \ldots, a_n \in A$, we have:

$$\mu_n(a_1, \ldots, a_n) = 0 \quad \text{if any } a_i = 1_A.$$

If A and B are unital A_∞^{pp}-algebras, a $\overset{pp}{\infty}$-morphism $f : A \rightsquigarrow B$ is said to be *unital* if

$$f_1(u_A) = u_B,$$

and

$$f_n^{p,q}(\mathrm{id}^{\otimes(i-1)} \otimes u_A \otimes \mathrm{id}^{\otimes(n-i)}) = 0$$

for all $n \geq 2$, $-n \leq p + q \leq 1 - n$, and $1 \leq i \leq n$.

That is, $f_1(1_A) = 1_B$, and for all $n > 1$, all $-n \leq p + q \leq 1 - n$, and all $a_1, \ldots, a_n \in A$, we have:

$$f_n^{p,q}(a_1, \ldots, a_n) = 0 \quad \text{if any } a_i = 1_A.$$

Example 2.12.2. A *unital associative algebra* is a unital A_∞^{pp}-algebra such that $\mu_n^{p,q} = 0$ for all $n \geq 3$.

Lemma 2.12.3. *Let $(A, \partial, \bar{\partial}, \mu)$ be a unital bidifferential bigraded algebra together with a contraction from A to a bicomplex B*

$$\begin{tikzcd} (A, \partial, \bar{\partial}) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (B, \partial, \bar{\partial}) \arrow[l, shift left, "g"] \end{tikzcd}$$

such that the following conditions are satisfied:

$$f \circ h = 0, \quad h \circ g = 0, \quad h \circ h = 0, \quad h\partial h = 0, \quad h\bar{\partial}h = 0 \quad \text{and} \quad h(1_A) = 0. \tag{2.5}$$

Then, the maps defined in Lemma 2.10.4 for all $n \geq 2$

$$\nu_n^{p,q} := f\mathfrak{m}_n^{p,q} g^{\otimes n} \quad \text{and} \quad g_n^{p,q} := \sum_{\substack{i-\alpha=p \\ j-\beta=q}} (-1)^{(\alpha+\beta)(i+j+n)} h_{\alpha\beta}\mathfrak{m}_n^{i,j} g^{\otimes n}$$

define

1. *a unital A_∞^{pp}-algebra $\left(B, \partial, \bar\partial, \{\nu_\bullet^{*,*}\}\right)$ with unit $1_B = f(1_A)$,*

2. *and a unital $\overset{pp}{\infty}$-weak equivalence $g = (g, \{g_\bullet^{*,*}\}) : B \rightsquigarrow A$.*

Proof. Recall that we denote $h_{10} := [\bar\partial, h]$ and $h_{01} := -[\partial, h]$. From Remark 1.2.3, the side conditions 2.5 imply

$$f \circ h_{10} = 0, \quad h_{10} \circ g = 0, \quad h_{10} \circ h_{10} = 0 \quad \text{and} \quad h_{10}(1_A) = 0,$$

$$f \circ h_{01} = 0, \quad h_{01} \circ g = 0, \quad h_{01} \circ h_{01} = 0 \quad \text{and} \quad h_{01}(1_A) = 0,$$

$$h \circ h_{10} = h_{10} \circ h = 0, \quad h \circ h_{01} = h_{01} \circ h = 0, \quad \text{and} \quad h_{10}h_{01} = -h_{01}h_{10} = h.$$

First, we observe that $\partial 1_B = \partial f 1_A = f \partial 1_A = 0$ and, in the same way, $\bar\partial 1_B = 0$. Let us show now that $g_1(1_B) = 1_A$. Indeed,

$$g_1(1_B) = gf(1_A) = 1_A + \partial\bar\partial h(1_A) - \partial h \bar\partial(1_A) + \bar\partial h \partial(1_A) - h\bar\partial\partial(1_A) = 1_A.$$

Recall that, by Remark 2.10.5, we can rewrite $\nu_n^{p,q}$ and $g_n^{p,q}$ as

$$\nu_n^{p,q} = \sum_{k+\ell=n} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} (-1)^{k(\ell+1)+d_1(d_2+\ell+1)} f\mu(g_k^{p_1,q_1} \otimes g_\ell^{p_2,q_2}),$$

and

$$g_n^{p,q} = \sum_{k+\ell=n} \sum_{\substack{p_1+p_2-\alpha=p \\ q_1+q_2-\beta=q}} (-1)^{k(\ell+1)+d_1(d_2+\ell+1)+(\alpha+\beta)(d_1+d_2+k+\ell)} h_{\alpha\beta}\mu(g_k^{p_1,q_1} \otimes g_\ell^{p_2,q_2}),$$

where $d_x = p_x + q_x$ for $x = 1, 2$. We will proceed by induction to prove

$$g_n^{p,q}(b_1, \dots, b_n) = 0$$

for all $n \geq 2$ if $b_j = 1_B$ for some $j = 1, \dots, n$. For $n = 2$ we have

$$g_2^{p,q}(b, 1_B) = h_{-p-q}\mu(g(b), 1_A) = h_{-p-q}g(b) = 0.$$

For $n > 2$, $g_n^{p,q}(b_1, \dots, b_n)$ is zero by induction hypothesis except if $k = 1$ and $j = 1$ or $\ell = 1$ and $j = n$. In the former case,

$$\begin{aligned} g_n^{p,q}(1_B, \dots, b_n) &= \sum_{\substack{p_2-\alpha=p \\ q_2-\beta=q}} (-1)^\varepsilon h_{\alpha\beta}\mu(1_A, g_{n-1}^{p_2,q_2}(b_2, \dots, b_n)) = \\ &= \sum_{\substack{p_2-\alpha=p \\ q_2-\beta=q}} (-1)^\varepsilon h_{\alpha\beta} g_{n-1}^{p_2,q_2}(b_2, \dots, b_n) = \\ &= \sum_{k+\ell=n-1} \sum_{\substack{p_1'+p_2'-\alpha'-\alpha=p \\ q_1'+q_2'-\beta'-\beta=q}} (-1)^\varepsilon h_{\alpha\beta}h_{\alpha'\beta'}\mu(g_k^{p_1',q_1'} \otimes g_\ell^{p_2',q_2'})(b_2, \dots, b_n). \end{aligned}$$

By the side conditions 2.5, the summands in the last expression are zero except for the cases

$(\alpha,\beta)=(1,0)$ and $(\alpha',\beta')=(0,1)$ or $(\alpha,\beta)=(0,1)$ and $(\alpha',\beta')=(1,0)$. So we have

$$\begin{aligned} g_n^{p,q}(1_B,\dots,b_n) = &\sum_{k+\ell=n-1}\sum_{\substack{p_1'+p_2'-1=p\\ q_1'+q_2'-1=q}} (-1)^{\varepsilon} h_{10}h_{01}\mu(g_k^{p_1',q_1'}\otimes g_\ell^{p_2',q_2'})(b_2,\dots,b_n)+\\ &+\sum_{k+\ell=n-1}\sum_{\substack{p_1'+p_2'-1=p\\ q_1'+q_2'-1=q}} (-1)^{\varepsilon} h_{01}h_{10}\mu(g_k^{p_1',q_1'}\otimes g_\ell^{p_2',q_2'})(b_2,\dots,b_n)=0, \end{aligned}$$

since $h_{10}h_{01}=-h_{01}h_{10}$. Similarly, one can prove the case $\ell=1$ and $j=n$:

$$g_n^{p,q}(b_1,\dots,1_B)=\sum_{\substack{p_1-\alpha=p\\ q_1-\beta=q}}(-1)^{\varepsilon}h_{\alpha\beta}\mu(g_{n-1}^{p_1,q_1}(b_2,\dots,b_n),1_A)=0.$$

Let us compute

$$\nu_2(b,1_B)=f\mu(g(b),1_A)=fg(b)=b$$

and

$$\nu_2(1_B,b)=f\mu(1_A,g(b))=fg(b)=b.$$

Finally, we show that

$$\nu_n^{p,q}(b_1,\dots,b_n)=0$$

for all $n\geq 3$ if $b_j=1_B$ for some $j=1,\dots,n$. Arguing similarly as for the maps $g_n^{p,q}$ we see that

$$\nu_n^{p,q}(b_1,\dots,b_n)$$

in non-zero only if $k=1$ and $j=1$ or $\ell=1$ and $j=n$, but both $\nu_n^{p,q}(1_B,\dots,b_n)$ and $\nu_n^{p,q}(b_1,\dots,1_B)$ lie in the image of $f\circ h_{\alpha\beta}$, and hence vanish by the side conditions. □

Of course, analogous definitions and results also hold in the commutative setting, which we now present.

Definition 2.12.4. A *unital C_∞^{pp}-algebra* is a unital A_∞^{pp}-algebra satisfying

$$\sum_{\sigma\in \mathrm{Sh}_{i,j}}\mathrm{sign}(\sigma)\mu_{i+j}^{p,q}\circ\sigma=0$$

for every $i,j\geq 1$ and every (p,q). Here, $\mathrm{Sh}_{i,j}$ denotes the set of (i,j)-shuffles.

A *unital $\overset{pp}{\infty}$-morphism of C_∞^{pp}-algebras $f:A\rightsquigarrow B$* is a unital $\overset{pp}{\infty}$-morphism of A_∞^{pp}-algebras satisfying the corresponding symmetric relations.

As a consequence of Lemma 2.12.3 and Lemma 2.11.3 we obtain

Lemma 2.12.5. *Let $(A,\partial,\bar\partial,\mu)$ be a unital commutative bidifferential bigraded algebra together with a contraction from A to a bicomplex B*

$$\begin{tikzcd} (A,\partial,\bar\partial) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (B,\partial,\bar\partial) \arrow[l, shift left, "g"] \end{tikzcd}$$

such that the following conditions are satisfied:

$$f \circ h = 0, \quad h \circ g = 0, \quad h \circ h = 0, \quad h\partial h = 0, \quad h\bar{\partial} h = 0 \quad \textit{and} \quad h(1_A) = 0. \tag{2.6}$$

Then, the maps defined in Lemma 2.10.4 for all $n \geq 2$

$$\nu_n^{p,q} := f\mathfrak{m}_n^{p,q} g^{\otimes n} \qquad \textit{and} \qquad g_n^{p,q} := \sum_{\substack{i-\alpha=p \\ j-\beta=q}} (-1)^{(\alpha+\beta)(i+j+n)} h_{\alpha\beta}\mathfrak{m}_n^{i,j} g^{\otimes n}$$

define

1. *a unital* C_∞^{pp}*-algebra* $\left(B, \partial, \bar{\partial}, \{\nu_\bullet^{*,*}\}\right)$ *with unit* $1_B = f(1_A)$,
2. *and a unital* $\overset{pp}{\infty}$*-weak equivalence of* C_∞^{pp}*-algebras* $g = (g, \{g_\bullet^{*,*}\}) : B \rightsquigarrow A$.

2.13 Cyclic A_∞^{pp}-algebras

The de Rham algebra of a closed smooth manifold carries a natural bilinear form by integrating forms of complementary degree. This pairing actually gives the Poincaré duality isomorphism. In this section, we study the interaction between A_∞^{pp}-algebras and this additional structure. More precisely, we introduce the notion of cyclic A_∞^{pp}-algebras and prove a homotopy transfer theorem for cyclic bidifferential bigraded algebras. This allows us to incorporate Poincaré duality to the real homotopy type of the manifold. Cyclic A_∞-algebras were introduced by Kontsevich in [Kon93], [Kon94], and further developed in [KS09]. The existence of minimal models for general cyclic algebras can be found in [CL09] and [CL10], however, we will adapt more ad hoc proofs for A_∞-algebras in [Kaj07] and [Laz06].

We will be considering bicomplexes $(B, \partial, \bar{\partial})$ endowed with a symmetric bilinear form

$$\langle\, , \rangle \colon B \times B \longrightarrow \mathbf{k}$$

such that:

- it is compatible with the differentials

$$\langle \partial x, y\rangle + (-1)^{|x|}\langle x, \partial y\rangle = 0 \quad \text{and} \quad \langle \bar{\partial} x, y\rangle + (-1)^{|x|}\langle x, \bar{\partial} y\rangle = 0,$$

- it is *homogeneous of bidegree* (d_1, d_2),

$$\langle x, y\rangle = 0 \quad \text{if } |x| + |y| \neq (d_1, d_2),$$

- and *symmetric*,

$$\langle x, y\rangle = (-1)^{|x||y|}\langle y, x\rangle.$$

Definition 2.13.1. A *cyclic contraction* of a bicomplex with bilinear form $(A, \partial, \bar{\partial}, \langle\, , \rangle)$ consists in a contraction into a bicomplex B, that is, a diagram

$$\begin{tikzcd} (A, \partial, \bar{\partial}, \langle\, , \rangle) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (B, \partial, \bar{\partial}), \arrow[l, shift left, "g"] \end{tikzcd}$$

where f and g are morphisms of bicomplexes satisfying $fg = \mathrm{id}_B$, and h is a *pluripotential homotopy* from the identity to gf, i.e.,

$$\mathrm{id}_A - gf = [\partial, [\bar{\partial}, h]].$$

Moreover, the contraction is required to be compatible with the bilinear form in the sense that

$$\langle hx, y\rangle = \langle x, hy\rangle \quad \text{and} \quad \langle gfx, y\rangle = \langle x, gfy\rangle. \tag{2.7}$$

Remark 2.13.2. Note that if (f, g, h) is a cyclic contraction, the chain homotopies given by

$$h_{10} := [\bar{\partial}, h] \quad \text{and} \quad h_{01} := -[\partial, h]$$

satisfy

$$\langle h_{10}x, y\rangle = (-1)^{|x|}\langle x, h_{10}y\rangle \quad \text{and} \quad \langle h_{01}x, y\rangle = (-1)^{|x|}\langle x, h_{01}y\rangle.$$

Definition 2.13.3. A *cyclic A_∞^{pp}-algebra* is a A_∞^{pp}-algebra $(A, \partial, \bar{\partial}, \{\mu_\bullet^{*,*}\})$ together with a homogeneous symmetric bilinear form such that

$$\langle \partial a, b\rangle + (-1)^{|a|}\langle a, \partial b\rangle = \langle \bar{\partial} a, b\rangle + (-1)^{|a|}\langle a, \bar{\partial} b\rangle = 0$$

for all $a, b \in A$ and

$$\langle \mu_n^{p,q}(a_1, \ldots, a_n), a_0\rangle = (-1)^{|a_0|(\sum_{i=1}^n |a_i|)+n}\langle \mu_n^{p,q}(a_0, \ldots, a_{n-1}), a_n\rangle$$

for all $n \geq 2$, $1 - n \leq p + q \leq 2 - n$ and $a_0, \ldots, a_n \in A$.

Example 2.13.4. A *cyclic bidifferential bigraded algebra* is a cyclic A_∞^{pp}-algebra such that $\mu_n^{p,q} = 0$ for all $n \geq 3$. Explicitly,

$$\langle \mu(a_1, a_2), a_0\rangle = \langle a_1, \mu(a_1, a_0)\rangle$$

for all $a_0, a_1, a_2 \in A$. The de Rham algebra of a closed complex manifold is a cyclic bba with pairing given by integration of forms

$$\langle \alpha, \beta\rangle = \int_M \alpha \wedge \beta,$$

for α a complex (p, q)-form and β a complex $(n - p, n - q)$-form and zero otherwise.

Definition 2.13.5. Let $(A, \langle\,,\rangle_A)$ and $(B, \langle\,,\rangle_B)$ be two cyclic A_∞^{pp}-algebras. A *cyclic $\overset{pp}{\infty}$-morphism* $f\colon A \rightsquigarrow B$ is a $\overset{pp}{\infty}$-morphism such that

$$\langle a_1, a_2\rangle_A = \langle f_1(a_1), f_1(a_2)\rangle_B$$

for any $a_1, a_2 \in A$, and for any $n \geq 3$ and $1 - n \leq i, j \leq 2 - n$

$$\sum_{k+\ell=n} \sum_{\substack{p+p'=i\\ q+q'=j}} \langle f_k^{p,q}(a_1, \ldots, a_k), f_\ell^{p',q'}(a_{k+1}, \ldots, a_n)\rangle_B = 0,$$

for all $a_1, \ldots, a_n \in A$.

We now state the homotopy transfer theorem for cyclic bba's.

Theorem 2.13.6. *Let $(A, \partial, \bar{\partial}, \langle\, ,\rangle)$ be a cyclic bidifferential bigraded algebra together with a cyclic contraction*

$$h \circlearrowright (A, \partial, \bar{\partial}, \langle\, ,\rangle) \underset{g}{\overset{f}{\rightleftarrows}} (B, \partial, \bar{\partial}).$$

Consider the maps defined in Lemma 2.10.4 for all $n \geq 2$

$$\nu_n^{p,q} := f\mathfrak{m}_n^{p,q} g^{\otimes n} \quad \text{and} \quad g_n^{p,q} := \sum_{\substack{i-\alpha=p \\ j-\beta=q}} (-1)^{(\alpha+\beta)(i+j+n)} h_{\alpha\beta}\mathfrak{m}_n^{i,j} g^{\otimes n}.$$

Then,

1. *With the induced pairing $\langle\, ,\rangle_B := \langle g, g\rangle$, the tuple $\left(B, \partial, \bar{\partial}, \{\nu_\bullet^{*,*}\}\right)$ is a cyclic A_∞^{pp}-algebra.*

2. *If the contraction satisfies the conditions $h \circ g$, $h \circ h = 0$, $h\partial h = 0$, and $h\bar{\partial}h = 0$, the $\overset{pp}{\infty}$-morphism $g = (g, \{g_\bullet^{*,*}\})\colon B \rightsquigarrow A$ is a cyclic $\overset{pp}{\infty}$-weak equivalence.*

Proof. We first prove 2. By Lemma 2.10.4, $g = (g, \{g_\bullet^{*,*}\})\colon B \rightsquigarrow A$ is a $\overset{pp}{\infty}$-weak equivalence, we have to prove cyclicity. On the one hand, the relation

$$\langle a_1, a_2\rangle_B = \langle g(a_1), g(a_2)\rangle$$

is true by definition. On the other, since the contraction is cyclic (Equations 2.7) and satisfies the side conditions, for $n \geq 3$ and $1 - n \leq i + j \leq 2 - n$ and for all $a_1, \ldots, a_n \in A$ we have

$$\sum_{k+\ell=n} \sum_{\substack{p+p'=i \\ q+q'=j}} \langle g_k^{p,q}(a_1, \ldots, a_k), g_\ell^{p',q'}(a_{k+1}, \ldots, a_n)\rangle =$$

$$= \sum_{k+\ell=n} \sum \pm\langle h_{\alpha\beta}\mathfrak{m}_k^{p,q}(a_1, \ldots, a_k), h_{\alpha'\beta'}\mathfrak{m}_\ell^{p',q'}(a_{k+1}, \ldots, a_n)\rangle = 0.$$

We also have $\langle g, g_{n-1}^{i,j}\rangle = \langle g_{n-1}^{i,j}, g\rangle = 0$.

To prove 1, note that, again by Lemma 2.10.4, $(B, \{\nu_\bullet^{*,*}\})$ is a A_∞^{pp}-algebra. It remains to prove the cyclicity, for which we adapt the proof given in [Laz06]. To simplify signs, we will work on suspended algebras. Let $s\colon B \to B[-1, 0]$ be the suspension isomorphism of bidegree $(-1, 0)$. Since $(B, \{\nu_\bullet^{*,*}\})$ is a A_∞^{pp}-algebra, sB is endowed with n-ary operations

$$\hat{\nu}_n^{p,q}\colon s\nu_n^{r,q}(s^{-1})^{\otimes n}\colon (sB)^{\otimes n} \longrightarrow sB$$

of total degree $0 \leq p + q \leq 1$ and $p = r + n - 1$ satisfying

$$\partial\hat{\nu}_n^{p,q} - (-1)^{p+q}\hat{\nu}_n^{p,q}\partial = q\hat{\nu}_n^{p+1,q} + \sum_{\substack{k+\ell=n+1 \\ 1\leq i\leq k}} \sum_{\substack{p_1+p_2=p+1 \\ q_1+q_2=q}} (-1)^{p_1+q_1+1}\hat{\nu}_k^{p_1,q_1}(\mathrm{id}^{i-1} \otimes \hat{\nu}_\ell^{p_2,q_2} \otimes \mathrm{id}^{k-i})$$

and

$$\bar{\partial}\hat{\nu}_n^{p,q} - (-1)^{p+q}\hat{\nu}_n^{p,q}\bar{\partial} = -p\hat{\nu}_n^{p,q+1} + \sum_{\substack{k+\ell=n+1\\ 1\le i\le k}} \sum_{\substack{p_1+p_2=p\\ q_1+q_2=q+1}} (-1)^{p_1+q_1+1}\hat{\nu}_k^{p_1,q_1}(\mathrm{id}^{i-1}\otimes\hat{\nu}_\ell^{p_2,q_2}\otimes\mathrm{id}^{k-i}).$$

The induced bilinear form $\langle , \rangle_B$ defines an antisymmetric bilinear form

$$\langle\, , \rangle\colon sB\otimes sB \longrightarrow sB \quad \text{given by} \quad \langle s^{-1}, s^{-1}\rangle_B.$$

Then $(B, \langle\, , \rangle_B, \{\nu_\bullet^{*,*}\})$ is a cyclic A_∞^{pp}-algebra if and only if

$$\langle\hat{\nu}_n^{p,q}(sb_1,\dots,sb_n), sb_0\rangle = (-1)^{(|b_0|+1)(|b_1+1+\cdots+|b_n|+1)}\langle\hat{\nu}_n^{p,q}(sb_0,\dots,sb_{n-1}), sb_n\rangle.$$

Since (f,g,h) is a cyclic contraction from A to B, we have that $(\hat{f} = sfs^{-1}, \hat{g} = sgs^{-1}, \hat{h} = shs^{-1})$ is a contraction from sA to sB such that

$$\langle\hat{h}(M), y\rangle = \langle x, \hat{h}(y)\rangle \quad \text{and} \quad \langle\hat{g}\hat{f}(M), y\rangle = \langle\hat{g}\hat{f}(M), y\rangle. \tag{2.8}$$

The formulas for the suspended maps $\hat{\mathfrak{m}}_n^{p,q} := s\mathfrak{m}_n^{r,q}(s^{-1})^{\otimes n}$ become

$$\hat{\mathfrak{m}}_n^{p,q} = \sum_{k+\ell=n} \sum_{\substack{p_1+p_2-\alpha_1-\alpha_2=p\\ q_1+q_2-\beta_1-\beta_2=q}} (-1)^{\alpha_1+\beta_1+1+\alpha_2+\beta_2+1}\hat{\mu}(\hat{h}_{\alpha_1\beta_1}\hat{\mathfrak{m}}_k^{p_1,q_1}\otimes\hat{h}_{\alpha_2\beta_2}\hat{\mathfrak{m}}_\ell^{p_2,q_2}),$$

and $\hat{\nu}_n^{p,q} = \hat{f}\hat{\mathfrak{m}}_n^{p,q}\hat{g}^{\otimes n}$.

We now define maps

$$P_{n+1}^{p,q}\colon (sB)^{\otimes n+1} \longrightarrow \mathbf{k}$$

given by

$$P_{n+1}^{p,q}(x_1,\dots,x_n,x_0) := \langle\hat{g}\hat{\nu}_n^{p,q}(x_1,\dots,x_n), \hat{g}x_0\rangle.$$

Let $\tau_{n+1}\colon (sB)^{\otimes n+1} \longrightarrow (sB)^{n+1}$ be the map given by

$$\tau_{n+1}(x_1\otimes\cdots\otimes x_n\otimes x_0) := (-1)^{|x_0|(|x_1|+\cdots+|x_n|)}x_0\otimes x_1\otimes\cdots\otimes x_n.$$

The cyclicity conditions then are equivalent to

$$P_{n+1}^{p,q}\circ\tau_{n+1} = P_{n+1}^{p,q}.$$

Therefore, to conclude the proof, it only remains to verify this identity. Note that conditions (2.8) imply

$$\langle\hat{g}\hat{\nu}_n^{p,q}(x_1,\dots,x_n), \hat{g}x_0\rangle = \langle\hat{\mathfrak{m}}_n^{p,q}(\hat{g}x_1,\dots,\hat{g}x_n), \hat{g}x_0\rangle.$$

One can give a graphical description of these maps in terms of trees. Recall from Remark 2.10.2 that the maps $\mathfrak{m}_n^{p,q}$ are obtained by summing over all planar binary trees whose internal edges are decorated with the three homotopies $h_{11} := h$, $h_{01} := -[\partial, h]$, and $h_{10} := [\bar{\partial}, h]$. The decorations must satisfy $\#h_{10} + \#h_{11} = p$ and $\#h_{01} + \#h_{11} = q$, where $\#h_{**}$ denotes the number of occurrences of the homotopy h_{**} in the tree. Let $PBT_n^{p,q}$ denote the collection of planar binary trees with n leaves such that:

- If $p+q=n-2$, the trees have exactly p internal edges labeled by h_{10} and q internal edges labeled by h_{01}.
- If $p+q=n-1$, the trees contain $p-1$ edges labeled by h_{10}, $q-1$ edges labeled by h_{01}, and a single edge labeled by h_{11}.

Then,

$$P_{n+1}^{p,q} = \sum_{t \in PBT_n^{p,q}} (-1)^{p+q+1} \langle \hat{\mathfrak{m}}_t \circ \hat{g}^{\otimes n}, \hat{g} \rangle,$$

where $\hat{\mathfrak{m}}_t$ is the operation representing the tree t using the homotopies $\hat{h}_{**}$. We can represent a summand $\langle \hat{\mathfrak{m}}_t \circ \hat{g}^{\otimes n}, \hat{g} \rangle$ as follows: we decorate every leaf of the tree t with $\hat{g}$. Every internal edge is decorated as $\hat{\mathfrak{m}}_t$. Now split the root by adding a new vertex (empty circle) in the middle. This creates two edges e', e'', where e'' is the new external edge. To e' we associate id and to e'' we associate the map $\hat{g}$, see Figure 2.1.

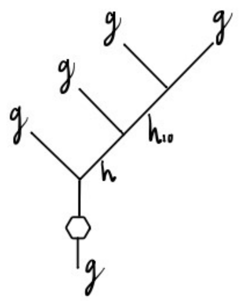


Figure 2.1: A tree representing a map $\langle \hat{\mathfrak{m}}_t^{p,q} \circ \hat{g}^{\otimes n}, \hat{g} \rangle$.

With this picture in mind, since the product on A is cyclic and the contraction verifies the Equations 2.7, the map

$$\langle \hat{\mathfrak{m}}_t \circ \hat{g}^{\otimes n}, \hat{g} \rangle$$

amounts to the map

$$\langle \hat{\mathfrak{m}}_{t'} \circ \hat{g}^{\otimes n}, \hat{g} \rangle \circ \tau_{n+1}$$

where t' is obtained from t by choosing the nth leaf to be the root, see Figure 2.2.

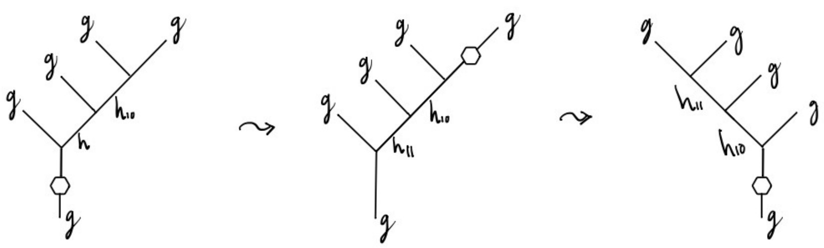


Figure 2.2: The process of "rerooting" a tree.

The process of "re-rooting" a planar binary tree in this way defines a map $PBT_n^{p,q} \longrightarrow PBT_n^{p,q}$ which is a bijection. This proves $P_{n+1}^{p,q} \circ \tau_{n+1} = P_{n+1}^{p,q}$. $\square$

Chapter 3

Higher structures in complex geometry

In this chapter, we explore several applications of Chapter 2 to complex manifolds. We study the connection between pluripotential A_∞-structures and certain Massey-like products defined for complex manifolds, known as ABC-*Massey products* [AT15], which are related to Bott-Chern and Aeppli cohomologies. Next, we show how the Pluripotential Homotopy Transfer Theorem can be employed to detect strong and weak formality [MS24], notions of formality associated with pluripotential weak equivalences. We test this approach on several examples.

Subsequently, we consider additional structure on the de Rham algebra of a compact oriented manifold, namely, the pairing given by integration of forms. This allows us to encode Poincaré duality on the real homotopy type of the manifold. We study the interaction of this pairing with pluripotential A_∞-structures in the Kähler setting. In particular, we give a concise proof of the strong formality of Kähler manifolds whose Hodge diamond corresponds to that of a complete intersection (cf. [Ste25a]).

3.1 Hodge decomposition

In this section, we review the definition of the Hodge star operator and state the Hodge Decomposition Theorem, which will be used in the subsequent sections. Most of the results of this section are classical and can be found, for instance, in [War83] or [WJ80].

Let M be a closed oriented Riemannian manifold of dimension n. The *Hodge star* operator

$$*\colon \mathcal{A}^*_{\mathrm{dR}}(M) \longrightarrow \mathcal{A}^{n-*}_{\mathrm{dR}}(M)$$

is defined so that it satisfies the following properties:

1. The Hodge star of the constant function 1 coincides with the Riemannian volume form: $*1 = \omega$.

2. The operator $*$ is linear.

3. For any smooth function f and differential form α, one has $*(f\alpha) = f(*\alpha)$.

4. If α is a p-form on an n-dimensional manifold, then $**\alpha = (-1)^{p(n-p)}\alpha$, so $\star$ is an isomorphism.

5. For every p-form α, we have $\alpha \wedge *\alpha = 0$ if and only if $\alpha = 0$.

6. If α and β are p-forms, then $\alpha \wedge *\beta = \beta \wedge *\alpha$.

To describe the local expression of the Hodge star, consider p-forms

$$\alpha = a_I \, dx^I, \qquad \beta = b_J \, dx^J,$$

where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_p)$, and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$.

Then the Hodge star operator satisfies

$$\alpha \wedge *\beta = \begin{cases} 0 & \text{if } I \neq J, \\ \pm a_I b_I \, \omega & \text{if } I = J, \end{cases}$$

where the sign is determined by the ordering of I and its complement.

In particular, for a p-form

$$\alpha = \sum a_I \, dx^I,$$

then

$$\alpha \wedge *\alpha = \left(\sum a_I^2\right) \omega.$$

The Hodge operator allows us to define an inner product on p-forms

$$(\ ,\ ) \colon \mathcal{A}^p_{\mathrm{dR}}(M) \times \mathcal{A}^p_{\mathrm{dR}}(M) \longrightarrow \mathbb{R},$$

for any p, by taking

$$(\alpha, \beta) := \int_M \alpha \wedge *\beta.$$

Define a differential

$$d^* \colon \mathcal{A}^p_{\mathrm{dR}} \to \mathcal{A}^{p-1}_{\mathrm{dR}} \quad \text{by} \quad d^* = (-1)^{np+n+1} * d *,$$

in degree p. By the properties of the Hodge star operator, we have $(d^*)^2 = 0$.

Proposition 3.1.1. *Let M be a closed oriented Riemannian manifold of dimension n. Let α and β be two forms. Then*

$$(d\alpha, \beta) = (\alpha, d^*\beta).$$

In particular, d and d^* are adjoint. This implies that closed forms under d are orthogonal to $\mathrm{Im}(d^*)$ and closed forms under d^* are orthogonal to $\mathrm{Im}(d)$.

The *Laplacian* is the degree-preserving linear map

$$\Delta \colon \mathcal{A}_{\mathrm{dR}}(M) \longrightarrow \mathcal{A}_{\mathrm{dR}}(M)$$

defined by

$$\Delta := dd^* + d^*d.$$

It is self adjoint, i.e. $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$.

Remark 3.1.2. A direct computation shows that for a form $\alpha \in \mathcal{A}_{\mathrm{dR}}$, we have $\Delta\alpha = 0$ if and only if $d\alpha = 0$ and $d^*\alpha = 0$.

Forms that are in the kernel of the Laplacian are called *harmonic* forms. We will denote the space of harmonic forms in degree k by

$$\mathcal{H}^k := \mathrm{Ker}(\Delta) \cap \mathcal{A}^k_{\mathrm{dR}}(M).$$

The following is a consequence of the fact that the Laplacian operator is elliptic, and uses deep results in functional analysis (see, for instance, [War83]).

Theorem 3.1.3 (Hodge Decomposition Theorem)**.** *Let M be a closed oriented Riemannian manifold. For each $k \geq 0$, there is an orthogonal direct sum decomposition*

$$\mathcal{A}^k_{dR}(M) = \mathcal{H}^k \oplus \mathrm{Im}(\Delta) = \mathcal{H}^k \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(d^*).$$

Define the Green operator

$$G : \mathcal{A}_{\mathrm{dR}}(M) \longrightarrow \mathcal{A}_{\mathrm{dR}}(M)$$

as $G(\mathcal{H}) = 0$ and $G = \Delta^{-1}$ in ImΔ. The Green operator G commutes with d, d^*, and Δ. In fact, G commutes with any linear operator that commutes with the Laplacian Δ.

Since closed forms under d are orthogonal to $\mathrm{Im}(d^*)$, we have that $\mathcal{H}$ and $\mathrm{Im}(d)$ give all closed forms under d in $\mathcal{A}_{\mathrm{dR}}$.

Theorem 3.1.4. *If M is a closed oriented Riemannian manifold, then*

$$\mathcal{H}^k \cong H^k_{dR}(M).$$

It is important to emphasize that the above theorem yields only isomorphisms of vector spaces, not of rings. In general, the space of harmonic forms is not closed under the wedge product.

From the Hodge decomposition, we obtain a canonical contraction of $\mathcal{A}_{\mathrm{dR}}(M)$ into the space of harmonic forms as follows [CW24]:

Proposition 3.1.5. *Let M be a closed oriented Riemannian manifold. There is a contraction*

$$\begin{tikzcd} \mathcal{A}_{\mathrm{dR}}(M) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & \mathcal{H} \arrow[l, shift left, hook, "g"] \end{tikzcd}$$

where f and g are given by the projection $\mathcal{A}_{\mathrm{dR}}(M) \twoheadrightarrow \mathcal{H}$ and inclusion $\mathcal{H} \hookrightarrow \mathcal{A}_{\mathrm{dR}}(M)$, respectively, and $h := d^*G$ is a chain homotopy from gf to the identity: $gf - \mathrm{id} = [d, h]$.

This chain contraction allows us, via the classical Homotopy Transfer Theorem, to endow the space of harmonic forms with an A_∞-algebra structure. As mentioned above, the space of harmonic forms is not closed under the wedge product. However, this procedure equips it with an associative product (note that the differential is trivial on $\mathcal{H}$).

3.2 Complex manifolds

In this section, we review some basic definitions concerning complex manifolds, with particular emphasis on their de Rham algebra.

Definition 3.2.1. A *complex manifold* (of complex dimension m) is a smooth manifold M with an open cover $M = \bigcup_i U_i$ and coordinate maps $\varphi_i \colon U_i \to \mathbb{C}^m$ such that the transition functions $\varphi_i \circ \varphi_j^{-1} : \mathbb{C}^m \to \mathbb{C}^m$ are holomorphic.

Remark 3.2.2. Every complex manifold of complex dimension m is a smooth manifold of real dimension $2m$. Note that while a smooth manifold can always be covered by open subsets diffeomorphic to $\mathbb{R}^n$, a general complex manifold cannot be covered by open subsets biholomorphic to $\mathbb{C}^n$. This phenomenon is due to the fact that $\mathbb{C}$ is not biholomorphic to a bounded open disc.

Some first examples of complex manifolds include $\mathbb{C}^n$, any open subset of $\mathbb{C}^n$, and $\mathbb{CP}^n$. We will later work with some more interesting examples.

Definition 3.2.3. A *holomorphic map* between complex manifolds $f \colon M \to N$ is a smooth map where the changes $\psi \circ f \circ \varphi^{-1}$ are holomorphic. A *biholomorphic map* is a bijective holomorphic map whose inverse is also holomorphic.

Let $U \subseteq \mathbb{C}^n$ be an open subset. In particular, it is a $2n$-dimensional smooth manifold, and so for each $x \in U$, T_xU has dimension $2n$. A canonical basis of T_xU is given by the tangent vectors

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n},$$

where $z_i = x_i + iy_i$ are the standard coordinates of $\mathbb{C}^n$. The vector space T_xU admits a *complex structure* I, given by

$$I\left(\frac{\partial}{\partial x_j}\right) = \left(\frac{\partial}{\partial y_j}\right) \text{ and } I\left(\frac{\partial}{\partial y_j}\right) = -\left(\frac{\partial}{\partial x_j}\right).$$

We can actually regard I as a bundle endomorphism $I \colon TU \to TU$ of the real tangent bundle of U.

Proposition 3.2.4. *Let $f \colon U \subset \mathbb{C}^n \to \mathbb{C}^n$ be a smooth map considered as a map from $\mathbb{R}^{2n}$ to itself. Then f is holomorphic if and only if the induced map*

$$f_* : T(\mathbb{C}^n) \longrightarrow T(\mathbb{C}^n)$$

is compatible with the complex structure, i.e., $f_* \circ I = I \circ f_*$.

Proposition 3.2.5. *The real tangent bundle TM of every complex manifold M is equipped with a linear endomorphism*

$$J : TM \longrightarrow TM \text{ such that } J^2 = -1.$$

Proof. It suffices to define J locally by $J = \varphi_{j*}^{-1} \circ I \circ \varphi_{j*}$, where $\varphi_j \colon U_j \to \mathbb{C}^n$ is a coordinate map. To see that this is well-defined, note that we have

$$\varphi_{j*}^{-1} \circ I \circ \varphi_{j*} = \varphi_{j*}^{-1} \circ I \circ \varphi_{ji*} \circ \varphi_{i*} = \varphi_{j*}^{-1} \circ \varphi_{ji*} \circ I \circ \varphi_{i*} = \varphi_{i*}^{-1} \circ I \circ \varphi_{i*}$$

where we used the fact that $\varphi_{ji} := \varphi_j \circ \varphi_i^{-1}$ is holomorphic and hence compatible with I. $\square$

Definition 3.2.6. An *almost complex structure* on a smooth manifold M is a linear endomorphism $J : TM \to TM$ such that $J^2 = -1$. The couple (M, J) is called *almost complex manifold.*

Remark 3.2.7. If M has an almost complex structure, then it must be even-dimensional. Let $v \in T_xM$, then Jv and v are linearly independent. Thus every vector generates a 2-dimensional subspace $\langle v, Jv\rangle$.

Definition 3.2.8. A *map of almost complex manifolds* is a smooth map $f\colon M \to N$ such that $J \circ f_* = f_* \circ J$.

Theorem 3.2.9 (Newlander-Nirenberg integrability theorem)**.** *An almost complex structure J on a smooth manifold M arises from the structure of a complex manifold if and only if $N_J \equiv 0$, where $N_J\colon TM \otimes TM \longrightarrow TM$ is the* Nijenhuis tensor

$$N_J(X,Y) := [X,Y] + J[X,JY] + J[JX,Y] - [JX,JY].$$

Remark 3.2.10. An easy exercise shows that any almost complex structure in a manifold of dimension 2 satisfies $N_J \equiv 0$. In particular, the above theorem implies that almost complex manifolds of dimension 2 are always complex curves. This is not true in higher dimensions. In real dimension 4, there exist almost complex manifolds that do not admit any integrable complex structure. A well-known example is the filiform nilmanifold. In real dimension ≥ 6, it is not known whether there exists an almost complex manifold that does not admit a complex structure. In particular, it is unknown whether the sphere S^6 admits the structure of a complex manifold (the known almost complex structures on S^6 are not integrable).

Example 3.2.11. Let $\mathfrak{g}$ be a nilpotent Lie algebra with the property that there exists a basis in $\mathfrak{g}$, $\{X_1, X_2, \ldots, X_n\}$, such that the constants c_k^{ij} arising in brackets

$$[X_i, X_j] = \sum_k c_k^{ij} X_k$$

are rational numbers for all i,j,k. Malcev proved in [Mal62] that there is a discrete co-compact subgroup Γ so that G/Γ is a compact smooth manifold, where $\mathfrak{g}$ is the Lie algebra of G. Manifolds constructed in this way are called *nilmanifolds.*

On $\Lambda\mathfrak{g}^*$, where $\mathfrak{g}^*$ denotes the dual of the vector space $\mathfrak{g}$, we can define a differential d given by the dual to the bracket on indecomposable elements,

$$dx_k(X_i, X_j) = -x_k([X_i, X_j]),$$

where $\{x_1, \ldots, x_n\}$ is the dual basis of $\mathfrak{g}^*$. As a consequence of Nomizu's theorem, the commutative differential graded algebra (cdga) $(\Lambda\mathfrak{g}^*, d)$ is a Sullivan minimal model of G/Γ.

Any almost complex structure on the Lie algebra $\mathfrak{g}$ (i.e. any $J : \mathfrak{g} \to \mathfrak{g}$ with $J^2 = -\mathrm{id}$) descends to a left invariant almost complex structure on the nilmanifold. This almost complex structure is integrable (i.e., it arises from a holomorphic atlas) if and only if $N_J = 0$ at the Lie algebra level.

Nilmanifolds provide a rich source of examples of complex manifolds. A main advantage is that they can be studied using finite-dimensional Lie algebra data. We will use some concrete examples of nilmanifolds, such as the Kodaira-Thurston manifold or the Iwasawa manifold, in Sections 3.4 and 3.5.

Let M be an almost complex manifold of dimension $2m$. Consider the complexification of its tangent bundle $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$. Then J extends to $T_{\mathbb{C}}M$ by $J(v \otimes z) := J(v) \otimes z$. Consider the $\pm i$-eigenspaces of J on $T_{\mathbb{C}}M$, given by:

$$T^{1,0}M := \{v \in T_{\mathbb{C}}M; J(v) = +iv\} \text{ and } T^{0,1}M := \{v \in T_{\mathbb{C}}M; J(v) = -iv\}.$$

This gives a direct sum decomposition

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M.$$

The complex-type decomposition of the tangent bundle also arises in the cotangent bundle $T^*_{\mathbb{C}}M := T^*M \otimes_{\mathbb{R}} \mathbb{C}$ as follows. Define $J\colon T^*_{\mathbb{C}}M \longrightarrow T^*_{\mathbb{C}}M$ by $J(\alpha(X)) := \alpha(J(X))$. It satisfies $J^2 = -\mathrm{id}$, and its $\pm i$-eigenspaces are given by

$$\Lambda^{1,0} := \{\alpha \in T^*_{\mathbb{C}}M; \alpha(Z) = 0 \; \forall Z \in T^{0,1}M\},$$

$$\Lambda^{0,1} := \{\alpha \in T^*_{\mathbb{C}}M; \alpha(Z) = 0 \; \forall Z \in T^{1,0}M\}.$$

Therefore,

$$T^*_{\mathbb{C}}M = \Lambda^{1,0} \oplus \Lambda^{0,1}.$$

Define $\Lambda^{p,q} := \Lambda^p(\Lambda^{1,0}) \otimes \Lambda^q(\Lambda^{0,1})$. This induces a bigrading on the algebra of differential forms with values in $\mathbb{C}$:

$$\mathcal{A}^n_{\mathrm{dR}}(M;\mathbb{C}) := \mathcal{A}_{\mathrm{dR}}(M)^n \otimes \mathbb{C} = \bigoplus_{p+q=n} A^{p,q},$$

where $A^{p,q}$ is by definition the space of sections of $\Lambda^{p,q}$.

Since $A^{p,q}$ is generated by $A^{0,0}$, $A^{0,1}$, and $A^{1,0}$, one may verify that for all $p, q \geq 0$, the exterior derivative satisfies

$$d(A^{p,q}) \subseteq A^{p-1,q+2} \oplus A^{p,q+1} \oplus A^{p+1,q} \oplus A^{p+2,q-1}.$$

Therefore, we may write $d = \bar{\mu} + \bar{\partial} + \partial + \mu$ where the bidegrees of each component are given by

$$|\bar{\mu}| = (-1,2),\; |\bar{\partial}| = (0,1),\; |\partial| = (1,0), \text{ and } |\mu| = (2,-1).$$

The components $\bar{\partial}$ and $\bar{\mu}$ are complex conjugate to ∂ and μ, respectively.

Expanding the equation $d^2 = 0$ we obtain the following set of equations:

$$\begin{aligned}
\mu^2 &= 0\\
\mu\partial + \partial\mu &= 0\\
\mu\bar{\partial} + \bar{\partial}\mu + \partial^2 &= 0\\
\mu\bar{\mu} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\mu}\mu &= 0\\
\bar{\mu}\partial + \partial\bar{\mu} + \bar{\partial}^2 &= 0\\
\bar{\mu}\bar{\partial} + \bar{\partial}\bar{\mu} &= 0\\
\bar{\mu}^2 &= 0
\end{aligned}$$

Also, by expanding the Leibniz rule,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge d\eta,$$

we obtain a Leibniz rule for each component of d. Note as well that, by degree reasons, $\bar{\mu}$ and μ vanish on functions, so both are linear over functions. In particular, $(\mathcal{A}^{*,*}, \bar{\mu})$ and $(\mathcal{A}^{*,*}, \mu)$ are commutative differential bigraded algebras.

Note that any map of almost complex manifolds $f\colon M \to M'$ induces a morphism of differential graded algebras $f^*\colon \mathcal{A}_{\mathrm{dR}}(M'; \mathbb{C}) \to \mathcal{A}_{\mathrm{dR}}(M; \mathbb{C})$ such that:

(i) f^* preserves the (p,q)-decompositions: $f^*(A^{p,q}_{M'}) \subseteq A^{p,q}_{M}$.

(ii) f^* is compatible with d component-wise: if δ is any of the components $\bar{\mu}$, $\bar{\partial}$, ∂ or μ of the differential d, then $f^*\delta = \delta f^*$.

The Nijenhuis tensor is encoded in the components μ and $\bar{\mu}$ of the exterior differential, thus giving

Theorem 3.2.12. *An almost complex manifold (M, J) is complex if and only if $d = \partial + \bar{\partial}$.*

In particular, the complexified de Rham complex of a complex manifold $\mathcal{A}_{\mathrm{dR}}(M; \mathbb{C})$ is a *bicomplex*. When considering the wedge product of forms, we obtain a *commutative bidifferential bigraded algebra.* We will denote by

$$H_{\partial}(M) := H_{\partial}(\mathcal{A}_{\mathrm{dR}}(M; \mathbb{C})) \quad \text{and} \quad H_{\bar{\partial}}(M) := H_{\partial}(\mathcal{A}_{\mathrm{dR}}(M; \mathbb{C})),$$

the anti-Dolbeault and the Dolbeault cohomologies. We also will denote by

$$H_{\mathcal{BC}}(M) := H_{\mathcal{BC}}(\mathcal{A}_{\mathrm{dR}}(M; \mathbb{C})) \quad \text{and} \quad H_{\mathcal{A}}(M) := H_{\mathcal{A}}(\mathcal{A}_{\mathrm{dR}}(M; \mathbb{C})),$$

the Bott-Chern and Aeppli cohomologies. Then, the ABC-cohomology of the manifold is given by $H_{\mathcal{ABC}}(M) = H_{\mathcal{BC}}(M) \oplus \operatorname{coker}(H_{\mathcal{BC}}(M) \to H_{\mathcal{A}}(M))$.

It is worth noting, as we did in Section 2.10, that the Dolbeault, anti-Dolbeault, and Bott–Chern cohomologies carry a natural algebra structure induced by the wedge product of forms. However, the Aeppli cohomology is only a module over the Bott–Chern cohomology. In particular, the ABC-cohomology is not an algebra in general. We will see later how we can endow this cohomology with a A_∞^{pp}-algebra structure.

3.3 Differential forms on Kähler manifolds

In this section, we briefly review some properties of an important class of complex manifolds, the so-called Kähler manifolds. For our purposes, we will mainly focus on their de Rham algebra.

Definition 3.3.1. A Riemannian metric g on a complex manifold (M, J) is called *Hermitian* if $g(JX, JY) = g(X, Y)$ for all $X, Y \in TM$. Its associated *fundamental form* is defined as

$$\omega(X, Y) := g(JX, Y).$$

Remark 3.3.2. Every complex manifold admits a Hermitian metric. Indeed, take a Riemannian metric g and define

$$h(X, Y) := g(X, Y) + g(JX, JY).$$

The fundamental 2-form, considered as a form with values in $\mathbb{C}$ is a $(1,1)$-form, that is, $\omega \in \mathcal{A}^{1,1}_{\mathrm{dR}}(M;\mathbb{C})$. If M is of complex dimension n, then ω^n is a volume form on M, in particular, M is oriented.

Definition 3.3.3. A *Kähler manifold* is a complex manifold with a Hermitian metric such that the associated fundamental form is closed: $dw = 0$.

Examples 3.3.4. First basic examples of Kähler manifolds include $\mathbb{C}^n$, $\mathbb{CP}^n$, and any smooth projective variety. Also, any complex curve is Kähler.

On a Kähler manifold, one obtains three operators d^*, ∂^* and $\bar\partial^*$ using the Hodge star operator by

$$d^* = - * d*, \qquad \partial^* = - * \partial *, \qquad \bar\partial^* = - * \bar\partial *,$$

where

$$d^* : A^m \longrightarrow A^{m-1}, \qquad \partial^* : A^{p,q} \longrightarrow A^{p-1,q}, \qquad \bar\partial^* : A^{p,q} \longrightarrow A^{p,q-1}.$$

The bicomplex equations for ∂ and $\bar\partial$ give bicomplex equations for ∂^* and $\bar\partial^*$: $(\partial^*)^2 = (\bar\partial^*)^2 = [\partial^*, \bar\partial^*] = 0$. The Laplacians associated with these operators are defined by

$$\Delta_d = dd^* + d^*d, \qquad \Delta_\partial = \partial\partial^* + \partial^*\partial, \qquad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

The space of harmonic forms with respect to Δ_δ is

$$\mathcal{H}^{p,q}_\delta = (\ker \Delta_\delta) \cap A^{p,q},$$

where δ denotes d, ∂ or $\bar\partial$. We can define as well Green operators G_δ, associated with Δ_δ.

Theorem 3.3.5. *If M is a Kähler manifold, we have*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial} \quad \textit{and} \quad [\partial, \bar\partial^*] = 0.$$

Remark 3.3.6. In fact, the converse was proven in [Oga70]. However, it is not known if the equality $[\partial, \bar\partial^*] = 0$, which follows from the first two, implies Kähler.

By Theoriem 3.3.5, Δ_d respect bidegrees, and Δ_∂ and $\Delta_{\bar\partial}$ are real operators. This fact, together with Theorem 3.1.4, implies the following.

Theorem 3.3.7. *Let M be a compact Kähler manifold. Then we have an isomorphism*

$$H^k_{\mathrm{dR}}(M) \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(M), \ \textit{and}\ H^{p,q}_{\bar\partial}(M) = \overline{H}^{q,p}_{\bar\partial}(M).$$

The above theorem gives topological obstructions for a compact manifold to be Kähler. First, we have

$$b_k = \sum_{p+q=k} h^{p,q} \text{ and } h^{p,q} = h^{q,p} = h^{n-q,n-p},$$

where $h^{p,q} := \dim H^{p,q}_{\bar\partial}(M)$ are the *Hodge numbers*. In particular, Betti numbers of odd degree are always even. A deeper result on the topology of compact Kähler manifolds is the formality theorem of Deligne, Griffiths, Morgan, and Sullivan [DGMS75]. They prove it by noting that the complexified de Rham algebra of a compact Kähler manifold satisfies the $\partial\bar{\partial}$-property (see Section 1.1). For readability, let us recall:

Definition 3.3.8. A complex manifold M is said to satisfy the *$\partial\bar{\partial}$-property* if

$$\mathrm{Ker}(\partial) \cap \mathrm{Im}(\bar{\partial}) = \mathrm{Ker}(\bar{\partial}) \cap \mathrm{Im}(\partial) = \mathrm{Im}(\partial\bar{\partial}).$$

Aside from Kähler manifolds, any compact complex manifolds admitting a proper Kähler modification also satisfies the $\partial\bar{\partial}$-property.

Theorem 3.3.9 ([DGMS75]). *Any compact complex manifold satisfying the $\partial\bar{\partial}$-property is formal in the sense of Sullivan: there are quasi-isomorphisms of cdga's*

$$\mathcal{A}_{\mathrm{dR}}(M) \xleftarrow{\sim} \bullet \xrightarrow{\sim} H^*(M),$$

where $H^*(M)$ is viewed as a cdga with trivial differential.

3.4 Massey products in complex geometry

In their study of the topology of complex manifolds, Angella and Tomasini [AT15] introduced triple Massey-type products for Bott–Chern and Aeppli cohomologies (called ABC-*Massey products*). In this section, we describe an explicit relation between these products and the A_∞^{pp}-algebra structure arising from the homotopy transfer theorem. The results of this section appear in [CGVSG25].

Let $(A, \partial, \bar{\partial}, \cdot)$ be a bidifferential bigraded algebra. Recall from Section 2.10 and Section 3.2 that $H_{\mathcal{BC}}(A)$ has an algebra structure induced by the product of the algebra and $H_{\mathcal{A}}(A)$ has the structure of a $H_{\mathcal{BC}}$-module.

We next recall the Aeppli-Bott-Chern-Massey (ABC-Massey) products defined in [AT15].

Definition 3.4.1. Let $[x], [y], [z] \in H_{\mathcal{BC}}(A)$ be three Bott-Chern cohomology classes such that $[x] \cdot [y] = [y] \cdot [z] = 0$. The *triple ABC-Massey product* $\langle [x], [y], [z] \rangle$ is defined as the following set of Aeppli classes:

$$\langle [x], [y], [z] \rangle := \{ [x \cdot v - u \cdot z] \mid \partial\bar{\partial}u = x \cdot y,\, \partial\bar{\partial}v = y \cdot z \} \subset H_{\mathcal{A}}(A).$$

The triple ABC-Massey product is said to be *trivial* if $0 \in \langle [x], [y], [z] \rangle$.

Let us remark that in [AT15] a different sign convention is used. We are sticking to the sign convention in [MS24] instead. Also, in these references, the triple product is defined as an element in the coset space

$$\frac{H_{\mathcal{A}}(A)}{[x] \cdot H_{\mathcal{A}}(A) + [z] \cdot H_{\mathcal{A}}(A)}$$

rather than as a set of Aeppli classes. Both viewpoints are equivalent. In particular, a triple ABC-Massey product is trivial if and only if it is zero in the above coset space.

When applied to the complexified de Rham algebra of a complex manifold, triple ABC-Massey products yield new holomorphic invariants with a homotopy–theoretic flavor. Moreover, in [MS24], these Massey-type operations were shown to obstruct the notions of *strong* and *weak formality* associated with pluripotential weak equivalences, see Section 3.5.

Interestingly, ABC-Massey products need not vanish in many situations where ordinary Massey products do. The standard proof of formality for compact Kähler manifolds of [DGMS75] relies on the $\partial\bar{\partial}$-property, but Sferruzza and Tomassini [ST22] constructed an example of a compact manifold satisfying the $\partial\bar{\partial}$-property while admitting a nontrivial ABC-Massey product (see Example 3.4.5). Subsequently, [PSZ24] showed that any closed Riemann surface of genus at least two carries a nontrivial ABC-Massey product. In particular, even compact Kähler manifolds may exhibit such nontrivial ABC-Massey products. The examples in [PSZ24] do not have restricted holonomy, leaving open the question of whether Calabi-Yau or hyperkähler manifolds are strongly formal. This was answered negatively for the Calabi-Yau case in the recent work [MMS25].

There are further contexts in which ABC-Massey products have been employed to obstruct the existence of certain metrics, for example in [ST23], [ST24], and [ST25]. Additional results on triple ABC-Massey products may be found in [TT17], [CT25], and [Rub25].

In [MS24], Milivojević and Stelzig introduce higher Aeppli-Bott-Chern Massey products, extending the notion of triple ABC-products, and prove that they are invariant under pluripotential weak equivalences. These higher ABC-Massey products take inputs and land in non-standard cohomology groups. Indeed, the possible targets of these products involve cohomology groups of the Bigolin complex (see Section 1.3) and their input and landing spaces depend on auxiliary choices. Moreover, they are not very accessible or computable in practice.

In contrast to these partially defined products, the homotopy transfer theorem for bidifferential bigraded algebras provides a robust homotopical framework for understanding higher products in cohomology. More precisely, in Section 2.10, we show that a minimal bicomplex (for instance, the $H_{\mathcal{ABC}}$-cohomology of a bba) can be endowed with a A_∞^{pp}-algebra structure that encodes the full homotopy type of the original algebra. In practice, this yields genuine operations, all acting on a single (typically small) space, which collectively capture all the relevant homotopical information. Moreover, [CGVSG25] presents a package ([GV24]) for computing such arity-3 products (which can be extended to higher arities), demonstrating that these operations are indeed computable.

We now explain the relation between triple ABC-Massey products and a transferred structure. Using Proposition 1.2.4, given a contraction into a minimal bicomplex $(H, \partial, \bar{\partial})$, we may define a new contraction

$$\begin{tikzcd} (A, \partial, \bar{\partial}) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (H_{\mathcal{ABC}}(A), \partial, \bar{\partial}), \arrow[l, shift left, "g"] \end{tikzcd} \tag{C}$$

where $f : A \twoheadrightarrow H \xrightarrow{\cong} H_{\mathcal{ABC}}(A)$ and $g : H_{\mathcal{ABC}}(A) \xrightarrow{\cong} H \hookrightarrow A$.

Applying the homotopy transfer theorem (Theorem 2.10.13) to this contraction yields an A_∞^{pp}-structure on $H_{\mathcal{ABC}}(A)$ together with a ${}^{pp}_{\infty}$-weak equivalence

$$H_{\mathcal{ABC}}(A) \rightsquigarrow A.$$

By Theorem 2.8.6, the transferred structure on $H_{\mathcal{ABC}}(A)$ encodes the homotopy type of A.

From now on, we will write $\mu_3 := \mu_3^{-1,-1}$, for the transferred arity 3 operation of bidegree $(-1, -1)$.

Proposition 3.4.2. *Given $[x],[y],[z] \in H_{\mathcal{BC}}(A)$ such that $[x]\cdot[y] = [y]\cdot[z] = 0$ and a contraction (f,g,h) as C, we have*

$$-[g\mu_3([x],[y],[z])]_{\mathcal{A}} \in \langle [x],[y],[z] \rangle,$$

where $[-]_{\mathcal{A}}$ denotes taking the Aeppli cohomology class.

Proof. Note first that there is an inclusion $H_{\mathcal{BC}}(A) \subseteq H_{\mathcal{ABC}}(A)$, so we may view such classes in ABC-cohomology. Write $x = g([x])$, $y = g([y])$ and $z = g([z])$. Since $[x]\cdot[y] = [y]\cdot[z] = 0$, we may choose an element u such that

$$\partial\bar{\partial}u = x \cdot y \text{ and } h(x\cdot y) = -u.$$

In the same way, we can choose v such that

$$\partial\bar{\partial}v = y \cdot z \text{ and } h(y\cdot z) = -v.$$

This gives the following

$$\mu_3([x],[y],[z]) = f(h(x\cdot y)\cdot z - x\cdot h(y\cdot z)) = f(-u\cdot z + x\cdot v).$$

Thus, we obtain

$$[g\mu_3([x],[y],[z])]_{\mathcal{A}} = [x\cdot v - u\cdot z]_{\mathcal{A}} \in \langle [x],[y],[z] \rangle.$$

□

Corollary 3.4.3. *Let $[x],[y],[z] \in H_{\mathcal{BC}}(A)$ such that $[x]\cdot[y] = [y]\cdot[z] = 0$. If $\mu_3([x],[y],[z]) = 0$ then the triple ABC-Massey product $\langle [x],[y],[z] \rangle$ is trivial.*

Remark 3.4.4. This reflects the situation observed in the ordinary case. Indeed, the triple products of A_∞-structures arising from homotopy transfer theory always detect Massey products. That is,

$$\pm\mu_3(a,b,c) \in \langle a,b,c \rangle.$$

(see, for example, [BMFM20]). Consequently, if $\mu_3(a,b,c) = 0$, the corresponding Massey product must also vanish.

In [BMFM20], an explicit relationship is established between A_∞-structures and ordinary higher Massey products. We expect that an analogous phenomenon should hold for pluripotential A_∞-algebras and higher ABC-Massey products. Although we do not develop this here, we believe it is an interesting direction for future work.

The following example exhibits the phenomenon described in Proposition 3.4.2. The computations presented here are extracted from [CGVSG25] and use the Sage package [GV24].

Example 3.4.5. The following procedure, explained in [ST22], gives a compact $\partial\bar{\partial}$-manifold by constructing an orbifold of global quotient type out of the Iwasawa nilmanifold.

Consider the action $\sigma\colon \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ given by $\sigma(z_1,z_2,z_3) = (iz_1, iz_2, -z_3)$. This gives a well-defined action on the Iwasawa manifold. The resulting quotient $M = \mathbb{I}_3/\langle\sigma\rangle$ is a compact orbifold with 16 isolated singular points. By Theorem 7.1 of [ST22], the orbifold M admits a resolution into a smooth $\partial\bar{\partial}$-compact manifold. The sub-algebra $\mathcal{A}_\sigma(M)$ of $\mathcal{A}(\mathbb{I}_3)$ of differential forms which are left invariant for the action of $\mathbb{H}(\mathbb{C})$ and σ-invariant is generated by the elements

$$a\bar{a}, b\bar{b}, c\bar{c}, a\bar{b}, b\bar{a}, c\bar{a}\bar{b}, ab\bar{c}, abc, \bar{a}\bar{b}\bar{c}.$$

The only non-trivial differentials being

$$\partial(c\bar{c}) = -ab\bar{c},\ \partial(c\bar{a}\bar{b}) = -ab\bar{a}\bar{b}$$

and their complex conjugates. This algebra satisfies the $\partial\bar{\partial}$-condition and so in particular, all cohomologies (including Dolbeault, Bott-Chern, Aeppli and ABC) coincide. In Table 3.1 it is indicated a basis for the cohomology in each bidegree.

$[\overline{abc}]$			$[abc\overline{abc}]$
		$[ac\overline{ac}]$, $[ac\overline{bc}]$, $[bc\overline{ac}]$, $[bc\overline{bc}]$	
	$[a\bar{a}]$, $[a\bar{b}]$, $[b\bar{a}]$, $[b\bar{b}]$		
1			$[abc]$

Table 3.1: A basis for the cohomology of M.

In [ST22], a non-trivial ABC-Massey product is computed on M:

$$\langle [a\bar{a}], [b\bar{b}], [b\bar{b}] \rangle_{ABC} \in \frac{H^{2,2}_{\mathcal{A}}(M)}{[a\bar{a}] \cdot H^{1,1}_{\mathcal{A}}(M) + [b\bar{b}] \cdot H^{1,1}_{\mathcal{A}}(M)},$$

which is represented by the nonzero Aeppli cohomology class $[bc\bar{b}\bar{c}] \in H^{2,2}_{A}(M)$.

We define contraction

$$\begin{tikzcd} (\mathcal{A}, \partial, \bar{\partial}) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & (H_{\mathcal{ABC}}(\mathcal{A}), \partial, \bar{\partial}) \arrow[l, shift left, "g"] \end{tikzcd}$$

as follows: the map g is induced by the identity on all elements. The map f is induced by the identity, except for the following elements, which belong to a square and for which f is zero:

$$\begin{tikzcd} c\bar{a}\bar{b} \arrow[r] & ab\bar{a}\bar{b} \\ c\bar{c} \arrow[r] \arrow[u] & ab\bar{c} \arrow[u] \end{tikzcd}$$

The homotopy h is given by

$$h(ab\bar{a}\bar{b}) = c\bar{c}$$

and zero otherwise.

Applying the homotopy transfer theorem we obtain that the μ_3-product is non-trivial only when all 3 inputs have bidegree $(1,1)$. We obtain

$$\mu_3([a\bar{a}], [b\bar{b}], [b\bar{b}]) = [b\bar{b}c\bar{c}]$$

which gives precisely the ABC-Massey product computed in [ST22].

3.5 Formality of complex manifolds

We next recall the two notions of weak and strong formality associated with pluripotential homotopy theory in the case of complex manifolds. Of course, any complex manifold M is a smooth manifold, and we may consider its real homotopy type, encoded in $\mathcal{A}_{\mathrm{dR}}(M)$. If this algebra is a formal commutative differential graded algebra (cdga), we will say that M is *de Rham formal*. Note that de Rham formality means that $\mathcal{A}_{\mathrm{dR}}(M)$ can be connected by a zigzag of quasi-isomorphisms of cdga's to a cdga whose underlying chain complex is minimal. Such an algebra is then isomorphic, as a cdga, to the de Rham cohomology of the manifold.

In the category $\mathrm{BiCo}_{\mathbf{k}}$ with pluripotential weak equivalences, minimality is equivalent to the condition $\partial\bar{\partial} = 0$. There is, however, a stronger property that also implies minimality, namely the vanishing of both differentials. This motivates the following two definitions [MS24].

Let M be a complex manifold and $\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C}) = \mathcal{A}_{\mathrm{dR}}(M) \otimes \mathbb{C}$ its complexified de Rham algebra.

Definition 3.5.1. A complex manifold M is called:

1. *Weakly formal* if $\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})$ is connected by a chain of pluripotential weak equivalences of cbba's to a cbba H satisfying $\partial_H\, \bar{\partial}_H = 0$.

2. *Strongly formal* if $\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})$ is connected by a chain of pluripotential weak equivalences of cbba's to a cbba H satisfying $\partial_H = \bar{\partial}_H = 0$.

Remark 3.5.2. Note that strong formality implies weak formality. Also, strong formality implies the $\partial\bar{\partial}$-property, and hence de Rham formality, so we may take H to be $H_{\mathcal{BC}}(M)$. Moreover, for $\partial\bar{\partial}$-manifolds, the notions of weak and strong formality coincide.

The following is proved in [MS24]. We will state the result only for triple products, but their results hold more generally.

Proposition 3.5.3. *On a weakly formal manifold, all triple ABC-Massey products vanish.*

By homotopy transfer (Theorem 2.11.4), there is a C_∞^{pp}-algebra $(H, \partial, \bar{\partial}, \{\nu_\bullet^{*,*}\})$ satisfying $\partial\bar{\partial} = 0$, and a $\overset{pp}{\infty}$-weak equivalence of C_∞^{pp}-algebras $g = (g, \{g_\bullet^{*,*}\})\colon H \rightsquigarrow \mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})$. This construction detects formality in the following sense:

Proposition 3.5.4. *If the operations $\nu_n^{p,q}$ vanish for all $n \geq 3$, then M is weakly formal. Moreover, if M satisfies the $\partial\bar{\partial}$-property, it is strongly formal.*

Proof. If $\nu_n^{p,q} = 0$ for $n \geq 3$, the bicomplex H with the transferred C_∞^{pp}-structure is a commutative bidifferential bigraded algebra. So we have two cbba's connected by a $\overset{pp}{\infty}$-weak equivalence of C_∞^{pp}-algebras. This means that the two algebras are isomorphic in the homotopy category $\mathrm{Ho}(\infty\text{-}C_\infty^{pp}\text{-Alg})$. By Theorem 2.8.6, we conclude that they are isomorphic in Ho(CBBA), where CBBA denotes the category of commutative bidifferential bigraded algebras. $\square$

We present two examples of formal complex manifolds from [MS24].

Example 3.5.5. The Kodaira-Thurston manifold KT is an example of a complex manifold that is not Kähler. It is defined as the quotient of the Lie group consisting of real matrices of the form

$$\begin{pmatrix} 1 & u_1 & u_3 & 0 & 0 \\ 0 & 1 & u_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & u_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

by the subgroup of matrices with integer entries. The Lie algebra of left-invariant vector fields is generated by V_1, V_2, V_3, V_4, with the only nontrivial bracket relation $[V_1, V_2] = -V_3$.

After dualizing, one obtains a Sullivan model for KT:

$$(\Lambda(v_1, v_2, v_3, v_4), \quad dv_3 = v_1 v_2),$$

given by the cdga of left-invariant one-forms. A left-invariant complex structure is given by $J(v_1) = v_2$ and $J(v_3) = v_4$. After complexification, define $x = v_1 - iv_2$ and $y = 2i(v_3 - iv_4)$. This yields a bigraded model of the complex manifold:

$$(\Lambda(x, \bar{x}, y, \bar{y}), \quad dy = x\bar{x}),$$

where x and y are of bidegree $(1, 0)$.

The natural inclusion of this bigraded algebra into the complex of differential forms on KT is a pluripotential weak equivalence. Indeed, it is a Dolbeault quasi-isomorphism [Rol11].

Note that this model satisfies $\partial\bar{\partial} = 0$, this shows that the Kodaira Thurston manifold is weakly formal.

This example illustrates that weak formality does not imply either the $\partial\bar{\partial}$-property or de Rham formality. Indeed, it is straightforward to see that this manifold does not satisfy the $\partial\bar{\partial}$-property. Moreover it is not de Rham formal, since it has a non-trivial (classical) Massey product given by $\langle x, \overline{x}, \overline{x}\rangle$.

Example 3.5.6. Let M be the Hopf surface obtained as the quotient

$$(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z},$$

where $\mathbb{Z}$ acts via scaling with a real constant $|\lambda|$ with $|\lambda| \neq 0, 1$. It is diffeomorphic to $S^3 \times S^1$. It is de Rham formal, however, it does not satisfy the $\partial\bar{\partial}$-property.

By [Ste25a, Lemma 3.10], the following cbba is pluripotentially weakly equivalent to the cbba of forms on M,

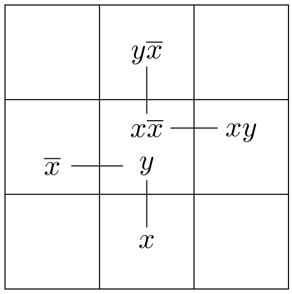
$$A = (\Lambda(x, \bar{x}, y, z, \partial z, \bar{\partial} z), dx = -d\bar{x} = y, \partial\bar{\partial} z = iy^2),$$

where M has bidegree $(1, 0)$ and both y and z have bidegree $(1, 1)$.

Recall from Section 1.1 that every bicomplex admits a (non-canonical) decomposition as the direct sum of a minimal bicomplex (i.e., one satisfying $\partial\bar{\partial} = 0$) and a bicomplex consisting entirely of squares. We decompose the bicomplex A as

$$A \cong H \oplus A_{\mathrm{sq}},$$

where H is the following minimal bicomplex



Following Proposition 1.2.4, we construct a contraction

$$h \circlearrowright (A, \partial, \bar{\partial}) \underset{g}{\overset{f}{\rightleftarrows}} (H, \partial, \bar{\partial}).$$

The map $f : A \twoheadrightarrow H$ and the map $g : H \hookrightarrow A$ are defined by the projection and inclusion, respectively.

Note that

$$H \cdot A_{\mathrm{sq}} \subseteq A_{\mathrm{sq}} \quad \text{and} \quad A_{\mathrm{sq}} \cdot A_{\mathrm{sq}} \subseteq A_{\mathrm{sq}}.$$

Moreover, by construction,

$$h(A) \subseteq A_{\mathrm{sq}}.$$

We now apply Lemma 2.11.3 to construct a C_∞^{pp}-structure $\{\nu_\bullet^{*,*}\}$ on H and a $\overset{pp}{\infty}$-weak equivalence $g\colon H \rightsquigarrow A$. Recall from Remark 2.10.5 that the C_∞^{pp}-structure is given by

$$\nu_n^{p,q} = \sum_{k+\ell=n} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} (-1)^\varepsilon f(g_k^{p_1,q_1} \cdot g_\ell^{p_2,q_2})$$

and

$$g_n^{p,q} = \sum_{k+\ell=n} \sum_{\substack{p_1+p_2-\alpha=p \\ q_1+q_2-\beta=q}} (-1)^\varepsilon h_{\alpha\beta}(g_k^{p_1,q_1} \cdot g_\ell^{p_2,q_2}),$$

for $n \geq 2$ where $h_{\alpha\beta}$ is either $h_{11} := h$, $h_{10} := [\bar{\partial}, h]$, or $h_{01} := -[\partial, h]$ and $g_1 = g$.

It follows that $\nu_n^{*,*} = 0$ for all $n \geq 3$, and hence M is weakly formal.

Examples of strongly formal manifolds will be given in Section 3.6. To conclude, we summarize the results of this and the previous section in the diagram of implications of Figure 3.1, where gray arrows indicate confirmed non-implications. The initials on the gray arrows indicate the counterexamples to the corresponding non-implications: KT refers to the Kodaira-Thurston manifold, ST denotes the manifold constructed by Sferruzza and Tomassini in [ST22] (reviewed in Section 3.4), and PZS stands for any of the examples presented in [PSZ24], for instance, a closed Riemannian surface of genus at least 2.

3.6 Cyclic A_∞^{pp}-structures in Kähler geometry

We show that for a compact Kähler manifold, there is always a cyclic contraction of its complexified de Rham algebra into the space of harmonic forms. We then endow the space of harmonic forms with a cyclic pluripotential A_∞-structure and present an alternative proof of the strong

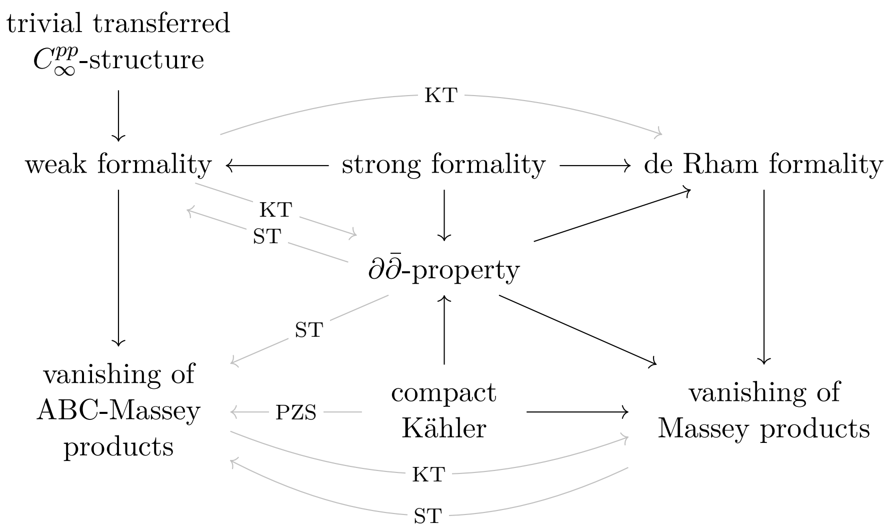


Figure 3.1: Diagram of implications

formality of Kähler manifolds whose Hodge diamond coincides with that of a complete intersection, as discussed in Section 3 of [Ste25a].

Let M be a compact oriented smooth manifold of dimension n. The de Rham algebra $\mathcal{A}_{\mathrm{dR}}(M)$ carries a graded symmetric bilinear form of degree n,

$$\langle -, - \rangle : \mathcal{A}_{\mathrm{dR}}(M) \times \mathcal{A}_{\mathrm{dR}}(M) \longrightarrow \mathbb{R},$$

defined by

$$\langle \omega, \eta \rangle := \int_M \omega \wedge \eta,$$

whenever $|\omega| + |\eta| = n$, and zero otherwise.

If M is a complex manifold of complex dimension n, we will also denote by $\langle\,,\rangle$ the linear extension of this bilinear form to the complexified de Rham algebra, which is a bilinear form of bidegree (n, n).

Remark 3.6.1. Note that this pairing is non-degenerate, since for any form $\alpha \in \mathcal{A}^k_{\mathrm{dR}}(M)$, we have

$$\langle \alpha, \star\alpha \rangle = (\alpha, \alpha) = 0 \text{ if and only if } \alpha = 0.$$

Here, $\star$ denotes the Hodge star operator. Moreover, let $\alpha, \beta \in \mathcal{A}^k_{\mathrm{dR}}(M; \mathbb{C})$. Note that, by definition,

$$\langle \star\alpha, \beta \rangle = (-1)^{k(n-k)} \langle \alpha, \star\beta \rangle.$$

This implies

$$\langle d^*\alpha, \beta \rangle = (-1)^{|\alpha|+1} \langle \alpha, d^*\beta \rangle,$$

and consequently,

$$\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle.$$

Here, d^* denotes the adjoint of d in the sense of Hodge theory, and Δ denotes the Laplacian, see Section 3.1.

Recall from Definition 2.13.1 that a contraction (f,g,h) of a bicomplex endowed with a bilinear form is said to be *cyclic* if the linear maps h and the composition gf are compatible with the pairing.

Proposition 3.6.2. *Let M be a compact Kähler manifold. Then, the complexified de Rham complex $\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})$ admits a cyclic contraction into the space of harmonic forms*

$$\begin{tikzcd} \mathcal{A}_{\mathrm{dR}}(M;\mathbb{C}) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & \mathcal{H} \arrow[l, shift left, "g"] \end{tikzcd}$$

satisfying the unital side conditions:

$$f\circ h=0,\quad h\circ g=0,\quad h\circ h=0,\quad h\partial h=0,\quad h\bar\partial h=0 \quad \textit{and} \quad h(1)=0.$$

Proof. By Hodge theory (Theorem 3.1.3) there is a decomposition

$$\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})=\mathcal{H}(M)\oplus \mathrm{Im}\Delta, \tag{3.1}$$

where Δ denotes the Laplacian with respect to d. Define

$$f:\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})\twoheadrightarrow\mathcal{H}(M) \qquad \text{and} \qquad g:\mathcal{H}(M)\hookrightarrow\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})$$

to be the obvious projection and injection. Now, we construct a pluripotential homotopy

$$h\colon \mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})\longrightarrow\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})$$

as

$$h=4\partial^*G\bar\partial^*G,$$

where G is the Green operator, see Section 3.1. By Theorem 3.3.5, we have $\Delta_\partial=\Delta_{\bar\partial}=\frac12\Delta$, $[\partial,\bar\partial^*]=0$ and $[\bar\partial,\partial^*]=0$. Moreover, ∂ and $\bar\partial$ commute with both the Green operator and the Laplacian. This implies

$$[\partial,[\bar\partial,\partial^*G\bar\partial^*G]]=-[\partial,\partial^*G\Delta_{\bar\partial}G]=-\Delta_\partial G\Delta_{\bar\partial}G.$$

In view of the splitting 3.1, we can write every form η in $\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})$ as $\eta_{\mathcal{H}}+\Delta b$, and we obtain

$$[\partial,[\bar\partial,h]](\eta_{\mathcal{H}}+\Delta b)=-4\Delta_\partial G\Delta_{\bar\partial}G(\eta_{\mathcal{H}}+\Delta b)=-4\Delta_\partial G(\frac{1}{2}\Delta b)=-\Delta(b)=(gf-\mathrm{id})(\eta_{\mathcal{H}}+\Delta b).$$

Now we see that it is compatible with the bilinear form $\langle\,,\rangle\colon\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})\times\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})\to\mathbb{C}$. On the one hand,

$$\langle gfx,y\rangle=\langle x,gfy\rangle$$

since the decomposition 3.1 is orthogonal with respect to the bilinear form. On the other hand, by Remark 3.6.1, ∂^*, $\bar\partial^*$ and G satisfy

$$\langle Gx,y\rangle=\langle x,Gy\rangle,\quad \langle\partial^*x,y\rangle=(-1)^{|x|+1}\langle x,\partial^*y\rangle \quad\text{and}\quad \langle\bar\partial^*x,y\rangle=(-1)^{|x|+1}\langle x,\bar\partial^*y\rangle$$

which implies
$$\langle hx, y\rangle = \langle x, hy\rangle.$$
Finally, we show that the unital side conditions are satisfied. Note that $h(1) = 0$ is satisfied by degree reasons. Since $fG = Gg = 0$ and $\partial^*\partial^* = 0$,
$$f \circ h = 0, \quad h \circ g = 0 \quad \text{and} \quad h \circ h = 0.$$
Now,
$$h\partial h = 16\partial^* G\bar{\partial}^* G\partial\partial^* G\bar{\partial}^* G = 0$$
since $[\partial, \bar{\partial}^*] = 0$ and $\bar{\partial}^*\bar{\partial}^* = 0$. Similarly, we obtain $h\bar{\partial}h = 0$ which completes the proof. □

Remark 3.6.3. Note that the restriction of the bilinear form to the space of harmonic forms is non-degenerate as well, since the Hodge decomposition is orthogonal with respect to the total bilinear form.

The contraction of Proposition 3.6.2 satisfies the conditions of Lemma 2.12.5 and Theorem 2.13.6, hence we obtain the following result

Theorem 3.6.4. *Let M be a compact Kähler manifold and let $(\mathcal{H}, \langle\, , \rangle)$ be the space of harmonic forms endowed with the restriction of the bilinear form*
$$\langle\, , \rangle \colon \mathcal{A}_{\mathrm{dR}}(M;\mathbb{C}) \times \mathcal{A}_{\mathrm{dR}}(M;\mathbb{C}) \to \mathbb{C}.$$
There is a unital cyclic C_∞^{pp}-structure $(\mathcal{H}, \langle\, , \rangle, \{\nu_\bullet^{*,*}\})$ and a unital cyclic C_∞^{pp}-weak equivalence of C_∞^{pp}-algebras $g \colon \mathcal{H} \rightsquigarrow \mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})$.

Remark 3.6.5. Note that, since the differentials ∂ and $\bar{\partial}$ vanish on the space $\mathcal{H}$ of harmonic forms of a complex manifold, by Remark 2.10.14, any C_∞^{pp}-structure $\{\nu_\bullet^{*,*}\}$ on $\mathcal{H}$ must satisfy $\nu_k^{p,q} = 0$ if $p + q = 2 - k$.

Proposition 3.6.6. *Let M be a compact Kähler manifold of complex dimension n and let $(\mathcal{H}, \langle\, , \rangle, \{\nu_\bullet^{*,*}\})$ be any unital cyclic C_∞^{pp}-structure on the space of harmonic forms. Then, the products $\nu_k^{p,q}(a_1, \ldots, a_k)$ for $k \geq 3$ landing in top bidegree (n, n) vanish.*

Proof. Suppose $\nu_k^{p,q}(a_1, \ldots, a_k)$ lands in bidegree (n, n). Since $\nu_k^{p,q}$ is cyclic and unital we have
$$\langle \nu_k^{p,q}(a_1, \ldots, a_n), 1\rangle = \langle \nu_k^{p,q}(1, a_1, \ldots, a_{n-1}), a_n\rangle = 0.$$
This implies $\nu_k^{p,q}(a_1, \ldots, a_n) = 0$. □

Let M be a compact Kähler manifold of complex dimension n. We say it has a Hodge diamond which matches that of a *complete intersection* if, denoting by $\delta = 1$ if $n/2 \in \mathbb{Z}$, and $\delta = 0$ otherwise, the Hodge numbers are given by:
$$h^{p,q}(M) = \begin{cases} 1 & \text{if } p = q \neq n/2, \\ d_{p,q} & \text{if } p + q = n \text{ and } p \neq q, \\ d_{p,q} + \delta & \text{if } p + q = n \text{ and } p = q, \\ 0 & \text{otherwise,} \end{cases}$$

where $d_{p,q}$ denotes the dimension of the space of primitive harmonic (p,q)-forms.

For example, the Hodge diamonds for $n=2$ and $n=3$ are:

For $n=2$**:**

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ d_{2,0} & & d_{1,1}+1 & & d_{0,2} \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

For $n=3$**:**

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ d_{3,0} & & d_{2,1} & & d_{1,2} & & d_{0,3} \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

There are many examples of such manifolds. Of course, every complete intersection is an example. But there are other examples, for instance, any Kähler surface with vanishing first Betti number.

Theorem 3.6.7 (cf. Theorem 3.5 in [Ste25a])**.** *Any compact Kähler manifold M of complex dimension $n \geq 2$ with the Hodge diamond of a complete intersection is strongly formal.*

Proof. We apply Theorem 3.6.4 to construct a unital cyclic C_∞^{pp}-structure on the space of harmonic forms $(\mathcal{H}, \langle\, , \rangle, \{\nu_\bullet^{*,*}\})$ and a $\overset{pp}{\infty}$-weak equivalence of C_∞^{pp}-algebras $g\colon \mathcal{H} \rightsquigarrow \mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})$. By Proposition 3.5.4, we just need to show that

$$\nu_k^{p,q} = 0 \quad \text{for all } k \geq 3.$$

By Remark 3.6.5, $\nu_k^{p,q} = 0$ if $p+q = 2-k$. Let $a_1, \dots, a_k \in \mathcal{H}$ and $k \geq 3$. First, note that if $a_i = \omega^{\ell_i}$ for all $i = 1, \dots, k$ where ω^{ℓ_i} denotes a power of the Kähler form, we have

$$\nu_k^{p,q}(a_1, \dots, a_k) = 0,$$

for all $k \geq 3$. Indeed, recall from Lemma 2.10.4 that the maps $\nu_n^{p,q}$ are defined as

$$\nu_n^{p,q} = \sum \pm f(h_{\alpha_1,\beta_1}(\mathfrak{m}_k^{p_1,q_1}) \cdot h_{\alpha_2,\beta_2}(\mathfrak{m}_\ell^{p_2,q_2}))g^{\otimes n}.$$

Here (f,g,h) is the contraction data given in Proposition 3.6.2. Recall also that $h_{\alpha\beta}$ is either h, $[\bar{\partial}, h]$ or $-[\partial, h]$. The maps $\mathfrak{m}_k^{i,j}$ are constructed using the wedge product in $\mathcal{A}_{\mathrm{dR}}(M;\mathbb{C})$ and the homotopies h, $[\bar{\partial}, h]$ or $-[\partial, h]$. Since h vanish on harmonic forms, we have $\mathfrak{m}_k^{i,j} = 0$ when applied to powers of the Kähler form for any $k \geq 3$.

Suppose now that there are r elements of $\{a_1, \dots a_k\}$ such that $a_j \neq \omega^{\ell_j}$. This implies $|a_j| = n$. Since the C_∞^{pp}-structure is unital, the lower bidegree of a non trivial product $\nu_k^{p,q}(a_1, \dots, a_k)$ is then reached when the rest of the elements are the Kähler form, of bidegree $(1,1)$. If $r \geq 2$

we have

$$|\nu_k^{p,q}(a_1, \dots, a_k)| \geq 2n + 2(k-2) - k + 1 = 2n + k - 3 \geq 2n \text{ for all } k \geq 3.$$

Therefore, $\nu_k^{p,q}(a_1, \dots, a_k)$ vanishes by Proposition 3.6.6.

If $r = 1$, then

$$\langle \nu_k^{p,q}(a_1, \dots, a_k), b \rangle = 0,$$

for all $b \in \mathcal{H}$. Indeed, for the pairing to be non-zero we need $|b| < n$, but then b must be a power of the Kähler form or a unit. Let $a_j \neq \omega^\ell$, we have

$$\langle \nu_k^{p,q}(a_1, \dots, a_k), b \rangle = \pm \langle \nu_k^{p,q}(a_{j+1}, \dots, b, \dots, a_{j-1}), a_j \rangle = 0,$$

either if b is a unit or a power of the Kähler form. □

Bibliography

[Aep65] Alfred Aeppli. On the cohomology structure of Stein manifolds. In *Proc. Conf. Complex Analysis (Minneapolis, 1964)*, pages 58–70. Springer, Berlin-Heidelberg-New York, 1965.

[AT15] Daniele Angella and Adriano Tomassini. On Bott-Chern cohomology and formality. *J. Geom. Phys.*, 93:52–61, 2015.

[BC65] Raoul Bott and Shiing-Shen Chern. Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections. *Acta Math.*, 114:71–112, 1965.

[Big69] Bruno Bigolin. Gruppi di Aeppli. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 23:259–287, 1969.

[BMFM20] Urtzi Buijs, José M. Moreno-Fernández, and Aniceto Murillo. A_∞-structures and Massey products. *Mediterr. J. Math.*, 17(1):Paper No. 31, 15, 2020.

[Bro73] Kenneth S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.*, 186:419–458, 1973.

[CG08] Xue Zhi Cheng and Ezra Getzler. Transferring homotopy commutative algebraic structures. *J. Pure Appl. Algebra*, 212(11):2535–2542, 2008.

[CGVSG25] Joana Cirici, Roger Garrido-Vilallave, and Anna Sopena-Gilboy. A_∞-structures for Bott-Chern and Aeppli cohomologies. *Complex Manifolds*, 12(1):Paper No. 20250012, 9, 2025.

[Cir15] Joana Cirici. Cofibrant models of diagrams: mixed Hodge structures in rational homotopy. *Trans. Amer. Math. Soc.*, 367(8):5935–5970, 2015.

[CL09] Joseph Chuang and Andrey Lazarev. Abstract Hodge decomposition and minimal models for cyclic algebras. *Lett. Math. Phys.*, 89(1):33–49, 2009.

[CL10] Joseph Chuang and Andrey Lazarev. Feynman diagrams and minimal models for operadic algebras. *J. Lond. Math. Soc., II. Ser.*, 81(2):317–337, 2010.

[CLW25] Joana Cirici, Muriel Livernet, and Sarah Whitehouse. Model category structures on truncated multicomplexes for complex geometry. *Bull. Lond. Math. Soc.*, 57(12):4163–4177, 2025.

[CR19] Joana Cirici and Agustí Roig. Sullivan minimal models of operad algebras. *Publ. Mat.*, 63(1):125–154, 2019.

[CSG22] Joana Cirici and Anna Sopena-Gilboy. Filtered A-infinity structures in complex geometry. *Proc. Amer. Math. Soc.*, 150(9):4067–4082, 2022.

[CT25] Nunzia Cesarino and Adriano Tomassini. Aeppli-Bott-Chern Massey products on non-Kähler solvmanifolds. Preprint, arXiv:2512.05894, 2025.

[CW24] Joana Cirici and Scott O. Wilson. Homotopy BV-algebras in Hermitian geometry. *Journal of Geometry and Physics*, 204:105275, 2024.

[DGMS75] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.

[Dol53] Pierre Dolbeault. Sur la cohomologie des variétés analytiques complexes. *C. R. Acad. Sci. Paris*, 236:175–177, 1953.

[Fre04] Benoit Fresse. Koszul duality of operads and homology of partition posets. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 115–215. Amer. Math. Soc., Providence, RI, 2004.

[GJ94] Ezra Getzler and John D. S. Jones. Operads, homotopy algebra and iterated integrals for double loop spaces. Preprint, arXiv:hep-th/9403055, 1994.

[Gom01] Robert E. Gompf. The topology of symplectic manifolds. *Turkish J. Math.*, 25(1):43–59, 2001.

[GV24] Roger Garrido-Vilallave. BiCo: Sage Script for Computing Invariants of Bigraded Complexes. https://github.com/geotop-ub/bico, 2024.

[HT90] Stephen Halperin and Daniel Tanré. Homotopie filtrée et fibrés C^∞. *Ill. J. Math.*, 34(2):284–324, 1990.

[Kad80] Tornike Kadeishvili. On the theory of homology of fiber spaces. *Uspekhi Mat. Nauk*, 35(3(213)):183–188, 1980.

[Kaj07] Hiroshige Kajiura. Noncommutative homotopy algebras associated with open strings. *Rev. Math. Phys.*, 19(1):1–99, 2007.

[Kon93] Maxim Kontsevich. Formal (non)-commutative symplectic geometry. In *The Gelfand Seminars, 1990-1992*, pages 173–187. Basel: Birkhäuser, 1993.

[Kon94] Maxim Kontsevich. Feynman diagrams and low-dimensional topology. In *First European congress of mathematics (ECM), Paris, France, July 6-10, 1992. Volume II: Invited lectures (Part 2)*, pages 97–121. Basel: Birkhäuser, 1994.

[KQ20] Mikhail Khovanov and You Qi. A faithful braid group action on the stable category of tricomplexes. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 16:Paper No. 019, 32, 2020.

[KS09] Maxim Kontsevich and Yan Soibelman. *Notes on A-infinty Algebras, A-infinity Categories and Non-Commutative Geometry*, pages 1–67. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[Laz06] Calin Iuliu Lazaroiu. Generating the superpotential on a D-brane category: I. Preprint, arXiv:hep-th/0610120, 2006.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic Operads*, volume 346 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer, Heidelberg, 2012.

[Mal62] Anatoli Ivanovich Malcev. On a class of homogeneous spaces. *Amer. Math. Soc. Translations Series 1*, 9:276–307, 1962.

[Mar06] Martin Markl. Transferring A_∞ (strongly homotopy associative) structures. In *The proceedings of the 25th winter school "Geometry and physics", Srní, Czech Republic, January 15–22, 2006*, pages 139–151. Palermo: Circolo Matemático di Palermo, 2006.

[MMS25] Lucía Martín-Merchán and Jonas Stelzig. Holomorphic linking numbers, ABC-Massey products, and Calabi-Yau 3-folds. Preprint, arXiv:2512.02187, 2025.

[MS24] Aleksandar Milivojević and Jonas Stelzig. Bigraded notions of formality and Aeppli-Bott-Chern-Massey products. *Comm. Anal. Geom.*, 32(10):2901–2933, 2024.

[MSG] Pedro Magalhães and Anna Sopena-Gilboy. The inflation functor in pluripotential homological algebra. *forthcoming.*

[MSS02] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in Algebra, Topology and Physics*, volume 96 of *Math. Surv. Monogr.* Providence, RI: American Mathematical Society (AMS), 2002.

[NT78] Joseph Neisendorfer and Laurence Taylor. Dolbeault homotopy theory. *Trans. Am. Math. Soc.*, 245:183–210, 1978.

[OEI26] OEIS Foundation Inc. OEIS sequence A003506. `https://oeis.org/A003506`, 2026. Accessed: 2026-01-16.

[Oga70] Yôsuke Ogawa. Operators on almost Hermitian manifolds. *J. Differ. Geom.*, 4:105–119, 1970.

[Pio25] Riccardo Piovani. On the cohomology of the Bigolin complex. Preprint, arXiv:2405.20823, 2025.

[PSZ24] Giovanni Placini, Jonas Stelzig, and Leopold Zoller. Nontrivial Massey products on compact Kähler manifolds. Preprint, arXiv:2404.09867, 2024.

[PSZ25] Giovanni Placini, Jonas Stelzig, and Leopold Zoller. Strong formality of toric and homogeneous compact Kähler manifolds. Preprint, arXiv:2510.17288, 2025.

[Roi22] Agustí Roig. Up-to-homotopy algebras with strict units. *J. Pure Appl. Algebra*, 226(6):Paper No. 106958, 31, 2022.

[Rol11] Sönke Rollenske. Dolbeault cohomology of nilmanifolds with left-invariant complex structure. In *Complex and differential geometry. Conference held at Leibniz Universität Hannover, Germany, September 14–18, 2009. Proceedings*, pages 369–392. Berlin: Springer, 2011.

[Rub25] Lapo Rubini. Some computations on trivial canonical-bundle solvmanifolds. *J. Geom. Phys.*, 216:Paper No. 105586, 30, 2025.

[Sch07] Michel Schweitzer. Autour de la cohomologie de Bott-Chern. Preprint, arXiv:0709.3528, 2007.

[SS03] Stefan Schwede and Brooke Shipley. Equivalences of monoidal model categories. *Algebr. Geom. Topol.*, 3:287–334, 2003.

[ST22] Tommaso Sferruzza and Adriano Tomassini. Dolbeault and Bott-Chern formalities: deformations and $\partial\overline{\partial}$-lemma. *J. Geom. Phys.*, 175:19, 2022. Id/No 104470.

[ST23] Tommaso Sferruzza and Adriano Tomassini. On cohomological and formal properties of strong Kähler with torsion and astheno-Kähler metrics. *Math. Z.*, 304(4):Paper No. 55, 27, 2023.

[ST24] Tommaso Sferruzza and Adriano Tomassini. Bott-Chern formality and Massey products on strong Kähler with torsion and Kähler solvmanifolds. *J. Geom. Anal.*, 34(11):Paper No. 348, 42, 2024.

[ST25] Tommaso Sferruzza and Adriano Tomassini. Hermitian geometrically formal manifolds. Preprint, arXiv:2507.10263, 2025.

[Sta63] James Dillon Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc.*, 108:293–312, 1963. **108** (1963), 275-292; ibid.

[Ste21] Jonas Stelzig. On the structure of double complexes. *J. Lond. Math. Soc. (2)*, 104(2):956–988, 2021.

[Ste25a] Jonas Stelzig. Pluripotential homotopy theory. *Adv. Math.*, 460:Paper No. 110038, 2025.

[Ste25b] Jonas Stelzig. Some remarks on the Schweitzer complex. *Ann. Inst. Fourier*, 75(1):35–47, 2025.

[TT17] Nicoletta Tardini and Adriano Tomassini. On geometric Bott-Chern formality and deformations. *Ann. Mat. Pura Appl. (4)*, 196(1):349–362, 2017.

[War83] Frank W. Warner. *Foundations of Differentiable Manifolds and Lie Groups. Reprint*, volume 94 of *Grad. Texts Math.* Springer, Cham, 1983.

[WJ80] Raymond O. Wells Jr. *Differential Analysis on Complex Manifolds. 2nd ed*, volume 65 of *Grad. Texts Math.* Springer, Cham, 1980.

[Yau24] Shing-Tung Yau. Open problems in geometry. *ICCM Not.*, 12(1):35–46, 2024.